University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 5

This problem set will be discussed on **Feb. 25**.

Problem 1: The Bloch sphere for mixed states

Any single-qubit density operator can be written as

$$\rho = \frac{\mathbb{1} + \vec{r} \cdot \vec{\sigma}}{2},$$

where $\vec{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$ with $\|\vec{r}\| \leq 1$ is the *Bloch vector*, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices.

a) Show that $\mathrm{Tr}(\rho^2) = \frac{1+\|\vec{r}\|^2}{2}$, and conclude that ρ is a pure state if and only if $\|\vec{r}\| = 1$. Hint[1].

b) Show that the Bloch vector components can be recovered from ρ via the formula

$$r_i = \mathrm{Tr}(\sigma_i \rho), \qquad i = x, y, z.$$

In other words: given ρ, this gives a systematic recipe for computing $\vec{r}$.

c) Use part (b) to compute the Bloch vector for each of the following density matrices and identify their positions in the Bloch ball:

- **i)** $\rho_1 = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$.
- **ii)** $\rho_2 = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$.

d) Going the other way: given a Bloch vector $\vec{r} = (r_x, r_y, r_z)$, write down the corresponding density matrix ρ. Use this to compute ρ for each of the following Bloch vectors:

- **i)** $\vec{r} = (0, 0, -1)$,
- **ii)** $\vec{r} = (1/2, 0, 0)$.

Identify which of these states are pure and which are mixed.

e) Consider the transformation $\rho \mapsto U\rho U^\dagger$ for some unitary U. Let $\vec{r}$ be the Bloch vector of ρ and $\vec{r}\,'$ be the Bloch vector of ρ'. Show that there is a real 3×3 matrix R such that $\vec{r}\,' = R\vec{r}$ and $R^T R = \mathbb{1}$. Conclude that the transformation $\rho \mapsto \rho'$ corresponds to a rotation of the Bloch vector. Hint[2].

f) A qubit has Bloch vector $\vec{r} = (r_x, r_y, r_z)$. Show that the probability of obtaining outcome $|0\rangle$ when measuring in the computational basis is

$$p(0) = \frac{1 + r_z}{2}.$$

Compute $p(0)$ and $p(1)$ for the state ρ_1 from part (c). What are the probabilities if the measurement is instead performed in the $\{|+\rangle, |-\rangle\}$ basis?

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[1] Use $\mathrm{Tr}(\sigma_i) = 0$, $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$, and $\mathrm{Tr}(\mathbb{1}) = 2$.

[2] Expand $U^\dagger \sigma_i U$ in the basis $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ and then use Hermiticity and trace to constrain the expansion coefficients.

Problem 2: Reduced Density Operators and the Partial Trace

a) Consider the two-qubit state $|\psi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$.

- **i)** Compute the density matrix $\rho_{AB} = |\psi\rangle\langle\psi|$.
- **ii)** Compute the reduced density operator $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$.
- **iii)** For what values of α and β is ρ_A maximally mixed (i.e. $\rho_A = \mathbb{1}/2$)?

b) Consider the maximally mixed state $\rho = \mathbb{1}/2$.

- **i)** Show that for any orthonormal basis $\{|e_0\rangle, |e_1\rangle\}$, the measurement probabilities on ρ are $p(0) = p(1) = \frac{1}{2}$.
- **ii)** Show that ρ is invariant under any unitary evolution: $U\rho U^\dagger = \rho$.
- **iii)** Using these results, explain why $\mathbb{1}/2$ contains no information and what is its Bloch vector? What does maximal entanglement imply about the information that can be extracted about the global state from either qubit alone?

c) The *GHZ state* is a three-qubit entangled state defined as:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big).$$

- **i)** Compute the reduced density operator ρ_{AB} obtained by tracing out qubit C.
- **ii)** Is ρ_{AB} a pure or mixed state? Is it entangled? If it is not entangled, explain in 1–2 sentences how it is possible that ρ_{AB} is not entangled while $|\mathrm{GHZ}\rangle$ is.

Problem 3: Schmidt Decomposition and Purification

This problem connects the Schmidt decomposition of bipartite pure states with reduced density operators and the purification of mixed states. Unless stated otherwise, all systems are qubits and we use the computational basis $\{|0\rangle, |1\rangle\}$.

a) General properties of the Schmidt decomposition and reduced states.

Let $|\psi\rangle_{AB}$ be a bipartite pure state with Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \lambda_i\ |i_A\rangle\,|i_B\rangle, \qquad \lambda_i > 0, \quad \sum_{i=1}^{r} \lambda_i^2 = 1,$$

and define the reduced states $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$.

- **i)** Prove that $|\psi\rangle$ is a product state if and only if its Schmidt number is $r = 1$.
- **ii)** Show that the Schmidt number is invariant under local unitaries, i.e. under $(U_A \otimes U_B)\,|\psi\rangle$.
- **iii)** Using the Schmidt decomposition, find expressions for ρ_A and ρ_B. What are their nonzero eigenvalues? Deduce that

$$\mathrm{rank}(\rho_A) = \mathrm{rank}(\rho_B) = r.$$

- **iv)** Explain in 2–4 sentences why for bipartite pure states, entanglement is reflected in the mixedness of reduced states.

v) Assume you have diagonalized ρ_A as

$$\rho_A = \sum_{i=1}^{r} p_i \, |i_A\rangle\langle i_A|, \qquad p_i > 0.$$

Define unnormalized states

$$|\widetilde{i}_B\rangle = (\langle i_A| \otimes \mathbb{1}_B)\,|\psi\rangle \in \mathcal{H}_B.$$

Show that $\| \, |\widetilde{i}_B\rangle\|^2 = p_i$ and that the normalized vectors

$$|i_B\rangle = \frac{1}{\sqrt{p_i}} \, |\widetilde{i}_B\rangle$$

form an orthonormal set in B. Conclude that

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \sqrt{p_i} \, |i_A\rangle \, |i_B\rangle$$

is a Schmidt decomposition.

This gives a systematic way of computing the Schmidt decomposition of any bipartite pure state: compute $\rho_A \mapsto$ diagonalize ρ_A to identify p_i and $|i_A\rangle \mapsto$ use the above method to find $|i_B\rangle \mapsto$ write down the Schmidt decomposition.

b) Schmidt decomposition computation.

Consider the two-qubit state

$$|\psi_2\rangle_{AB} = \frac{\sqrt{3}+1}{4}\big(|00\rangle + |11\rangle\big) + \frac{\sqrt{3}-1}{4}\big(|01\rangle + |10\rangle\big).$$

i) Compute $\rho_A = \mathrm{Tr}_B(|\psi_2\rangle\langle\psi_2|)$ in the computational basis $\{|0\rangle, |1\rangle\}$ of A.

ii) Diagonalize ρ_A: find its eigenvalues and an orthonormal eigenbasis.

iii) Use your diagonalization to find the Schmidt coefficients and Schmidt bases, and write the Schmidt decomposition of $|\psi_2\rangle$.

c) Purification as Schmidt decomposition "in reverse".

i) Construct the standard purification $|\Psi_{\mathrm{std}}\rangle_{AR}$ of the mixed state ρ_A you found in part (b) onto an ancilla system R.

ii) Verify that $\mathrm{Tr}_R(|\Psi_{\mathrm{std}}\rangle\langle\Psi_{\mathrm{std}}|) = \rho_A$.

iii) Identify $R \cong B$. Show that $|\Psi_{\mathrm{std}}\rangle_{AB}$ is generally not equal to the original state $|\psi_2\rangle_{AB}$ from the previous subproblem. Find an explicit unitary U_B such that

$$|\psi_2\rangle_{AB} = (\mathbb{1}_A \otimes U_B)\,|\Psi_{\mathrm{std}}\rangle_{AB}.$$

d) Consider a qubit state with Bloch vector $\vec{r} = (0, 0, r_z)$:

$$\rho = \frac{1}{2}\big(\mathbb{1} + r_z\sigma_z\big), \qquad -1 \le r_z \le 1.$$

i) Diagonalize ρ and construct a purification $|\Psi(r_z)\rangle_{AR}$ using the eigenbasis of ρ.

ii) Identify the Schmidt coefficients of $|\Psi(r_z)\rangle$ in terms of r_z, and state when the purification is maximally entangled and when it is a product state.

Problem 4: Teleportation and Reduced Density Operators

In this task you revisit the quantum teleportation protocol, but now you will keep track of Bob's state using reduced density operators.

Let the input qubit (to be teleported) be

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Alice holds two qubits A_1 (the input) and A_2 (her half of the Bell pair), and Bob holds B (his half of the Bell pair). Initially,

$$|\Psi_0\rangle = |\psi\rangle_{A_1} \otimes \left|\Phi^+\right\rangle_{A_2B}, \qquad \left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Alice applies a CNOT with control A_1 and target A_2, followed by a Hadamard on A_1. The resulting three-qubit state (just before Alice measures) can be written as

$$|\Psi_2\rangle = \frac{1}{2}\Big(|00\rangle_{A_1A_2}\,|\psi\rangle_B + |01\rangle_{A_1A_2}\,\sigma_x\,|\psi\rangle_B + |10\rangle_{A_1A_2}\,\sigma_z\,|\psi\rangle_B + |11\rangle_{A_1A_2}\,\sigma_x\sigma_z\,|\psi\rangle_B\Big).$$

a) Compute Bob's reduced density operator before Alice measures,

$$\rho^B_{\text{pre}} = \mathrm{Tr}_{A_1A_2}\big(|\Psi_2\rangle\langle\Psi_2|\big).$$

Show that $\rho^B_{\text{pre}} = \mathbb{1}/2$, independent of α, β. Hint[3].

b) Alice measures (A_1, A_2) in the computational basis and gets a two-bit outcome $m \in \{00, 01, 10, 11\}$.

- **i)** Show that $p(m) = \frac{1}{4}$ for all outcomes.
- **ii)** For each m, write Bob's (normalized) post-measurement density operator ρ^B_m.

c) Bob does not know Alice's measurement outcome yet. Compute Bob's state after Alice's measurement but before the classical message arrives,

$$\rho^B_{\text{after}} = \sum_m p(m)\,\rho^B_m.$$

Show that $\rho^B_{\text{after}} = \rho^B_{\text{pre}} = \mathbb{1}/2$. What does this imply about what Bob can learn *locally* at this stage?

d) After receiving $m = m_1m_2$, Bob applies the usual teleportation correction

$$U_m = \sigma_z^{m_1}\sigma_x^{m_2}.$$

Show (using density operators) that Bob then recovers the teleported state:

$$\rho^B_{\text{corr}} = |\psi\rangle\langle\psi|.$$

e) In 2–4 sentences, briefly summarize why the teleportation protocol does not allow Alice to signal to Bob faster than light. In particular, explain why Bob cannot learn anything about $|\psi\rangle$ (or even whether Alice has already measured) before the two classical bits m arrive.

[3] Write $|\Psi_2\rangle = \frac{1}{2}\sum_{m\in\{00,01,10,11\}} |m\rangle_{A_1A_2}\,|\phi_m\rangle_B$.

f) General proof of no-signaling.

Let ρ^{AB} be an arbitrary bipartite state (not necessarily pure), where A is Alice's entire system and B is Bob's system, and let $\rho^B = \mathrm{Tr}_A(\rho^{AB})$ be Bob's reduced state. Alice measures her system with general measurement operators $\{M_m^A\}$, satisfying completeness $\sum_m (M_m^A)^\dagger M_m^A = \mathbb{1}_A$. The measurement outcomes m have probabilities $p(m)$ and corresponding post-measurement states ρ_m^{AB}. Let $\rho_m^B = \mathrm{Tr}_A(\rho_m^{AB})$ be Bob's reduced state conditioned on outcome m. Show that Bob's reduced state, averaged over outcomes in the usual way, is unchanged by the measurement:

$$\sum_m p(m)\, \rho_m^B = \mathrm{Tr}_A(\rho^{AB}) = \rho^B.$$