University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 5

This problem set will be discussed on **Feb. 25**.

Problem 1: The Bloch sphere for mixed states

Any single-qubit density operator can be written as

$$\rho = \frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2},$$

where $\vec{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$ with $\|\vec{r}\| \leq 1$ is the *Bloch vector*, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices.

a) Show that $\mathrm{Tr}\big(\rho^2\big) = \frac{1+\|\vec{r}\|^2}{2}$, and conclude that ρ is a pure state if and only if $\|\vec{r}\| = 1$. Hint[1].

Solution: We compute:

$$\rho^2 = \frac{1}{4}\big(\mathbb{1} + \vec{r}\cdot\vec{\sigma}\big)^2 = \frac{1}{4}\Big(\mathbb{1} + 2\vec{r}\cdot\vec{\sigma} + (\vec{r}\cdot\vec{\sigma})^2\Big).$$

Using $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$, we get $(\vec{r}\cdot\vec{\sigma})^2 = \|\vec{r}\|^2\mathbb{1}$ (the cross terms cancel by antisymmetry of ϵ_{ijk}). Therefore:

$$\rho^2 = \frac{1}{4}\big((1 + \|\vec{r}\|^2)\mathbb{1} + 2\vec{r}\cdot\vec{\sigma}\big).$$

Taking the trace and using $\mathrm{Tr}(\mathbb{1}) = 2$ and $\mathrm{Tr}(\sigma_i) = 0$:

$$\mathrm{Tr}\big(\rho^2\big) = \frac{1}{4}\big(2(1 + \|\vec{r}\|^2) + 0\big) = \frac{1 + \|\vec{r}\|^2}{2}.$$

Since $\mathrm{Tr}\big(\rho^2\big) \leq 1$ with equality iff ρ is pure, we have ρ pure $\iff \|\vec{r}\|^2 = 1 \iff \|\vec{r}\| = 1$. Points strictly inside the Bloch ball ($\|\vec{r}\| < 1$) correspond to mixed states, points at the surface ($\|\vec{r}\| = 1$) correspond to pure states, and the center ($\vec{r} = \vec{0}$) is the maximally mixed state $\mathbb{1}/2$.

b) Show that the Bloch vector components can be recovered from ρ via the formula

$$r_i = \mathrm{Tr}(\sigma_i\rho), \qquad i = x, y, z.$$

In other words: given ρ, this gives a systematic recipe for computing $\vec{r}$.

Solution: Write

$$\rho = \frac{1}{2}\Big(\mathbb{1} + \sum_{j\in\{x,y,z\}} r_j\sigma_j\Big).$$

[1] Use $\mathrm{Tr}(\sigma_i) = 0$, $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$, and $\mathrm{Tr}(\mathbb{1}) = 2$.

Multiply by σ_i and take the trace:

$$\begin{aligned}\mathrm{Tr}(\sigma_i\rho) &= \frac{1}{2}\Big(\mathrm{Tr}(\sigma_i) + \sum_j r_j\ \mathrm{Tr}(\sigma_i\sigma_j)\Big) \\ &= \frac{1}{2}\Big(0 + \sum_j r_j \cdot 2\delta_{ij}\Big) = r_i.\end{aligned}$$

Hence the components are given by

$$r_i = \mathrm{Tr}(\sigma_i\rho), \qquad i = x, y, z.$$

c) Use part (b) to compute the Bloch vector for each of the following density matrices and identify their positions in the Bloch ball:

- **i)** $\rho_1 = \frac{3}{4}\,|0\rangle\langle 0| + \frac{1}{4}\,|1\rangle\langle 1|$.
- **ii)** $\rho_2 = \frac{1}{2}\,|+\rangle\langle +| + \frac{1}{2}\,|-\rangle\langle -|$.

Solution:

i) In matrix form,

$$\rho_1 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

Then $r_x = \mathrm{Tr}(\sigma_x\rho_1) = 0$ and $r_y = \mathrm{Tr}(\sigma_y\rho_1) = 0$ since σ_x, σ_y are off-diagonal while ρ_1 is diagonal. Moreover,

$$r_z = \mathrm{Tr}(\sigma_z\rho_1) = \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} 3/4 & 0 \\ 0 & -1/4 \end{pmatrix}\right) = \frac{1}{2}.$$

So $\vec{r}_1 = (0, 0, \frac{1}{2})$, i.e. ρ_1 lies on the positive z-axis, halfway between the center and the north pole.

ii) Using $|\pm\rangle\langle\pm| = \frac{1}{2}(\mathbb{1} \pm \sigma_x)$,

$$\rho_2 = \frac{1}{2}\cdot\frac{\mathbb{1} + \sigma_x}{2} + \frac{1}{2}\cdot\frac{\mathbb{1} - \sigma_x}{2} = \frac{\mathbb{1}}{2}.$$

Hence $r_i = \mathrm{Tr}(\sigma_i\rho_2) = 0$ for all i, so $\vec{r}_2 = (0, 0, 0)$: the maximally mixed state at the center of the Bloch ball.

d) Going the other way: given a Bloch vector $\vec{r} = (r_x, r_y, r_z)$, write down the corresponding density matrix ρ. Use this to compute ρ for each of the following Bloch vectors:

- **i)** $\vec{r} = (0, 0, -1)$,
- **ii)** $\vec{r} = (1/2, 0, 0)$.

Identify which of these states are pure and which are mixed.

Solution: From $\rho = \frac{1}{2}(\mathbb{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$ and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we obtain the general conversion formula

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}.$$

i) For $\vec{r} = (0, 0, -1)$,

$$\rho = \frac{1}{2}\begin{pmatrix} 1 - 1 & 0 \\ 0 & 1 - (-1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|,$$

which is pure since $\|\vec{r}\| = 1$.

ii) For $\vec{r} = (1/2, 0, 0)$,

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}.$$

This is mixed since $\|\vec{r}\| = 1/2 < 1$.

e) Consider the transformation $\rho \mapsto U\rho U^\dagger$ for some unitary U. Let $\vec{r}$ be the Bloch vector of ρ and $\vec{r}\,'$ be the Bloch vector of ρ'. Show that there is a real 3×3 matrix R such that $\vec{r}\,' = R\vec{r}$ and $R^T R = \mathbb{1}$. Conclude that the transformation $\rho \mapsto \rho'$ corresponds to a rotation of the Bloch vector. Hint[2].

Solution: Let $\rho' = U\rho U^\dagger$ and write $\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})$ and $\rho' = \frac{1}{2}(\mathbb{1} + \vec{r}\,'\cdot\vec{\sigma})$. Using the recipe from part (b),

$$r_i' = \mathrm{Tr}\big(\sigma_i\rho'\big) = \mathrm{Tr}\Big(\sigma_i U\rho U^\dagger\Big) = \mathrm{Tr}\Big(U^\dagger\sigma_i U\,\rho\Big). \tag{1}$$

Since $\mathrm{Tr}\big(U^\dagger\sigma_i U\big) = \mathrm{Tr}(\sigma_i) = 0$, the operator $U^\dagger\sigma_i U$ has no $\mathbb{1}$ component. It can therefore be expanded in the Pauli basis as

$$U^\dagger\sigma_i U = \sum_{j\in\{x,y,z\}} R_{ij}\sigma_j,$$

for some expansion coefficients R_{ij}. The expansion coefficients can be found by multiplying with σ_k and taking the trace:

$$\sigma_k U^\dagger\sigma_i U = \sigma_k \sum_{j\in\{x,y,z\}} R_{ij}\sigma_j \Rightarrow$$

$$\mathrm{Tr}\Big(\sigma_k U^\dagger\sigma_i U\Big) = \mathrm{Tr}\left(\sigma_k \sum_{j\in\{x,y,z\}} R_{ij}\sigma_j\right) = \sum_{j\in\{x,y,z\}} R_{ij}\,\mathrm{Tr}(\sigma_j\sigma_k) = 2\sum_{j\in\{x,y,z\}} R_{ij}\delta_{jk} = 2R_{ik}$$

Thus, (by relabeling $k \to j$):

$$R_{ij} = \frac{1}{2}\,\mathrm{Tr}\Big(\sigma_j U^\dagger\sigma_i U\Big)$$

[2] Expand $U^\dagger\sigma_i U$ in the basis $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ and then use Hermiticity and trace to constrain the expansion coefficients.

R_{ij} is real since for Hermitian A, B one has $\mathrm{Tr}(AB) \in \mathbb{R}$. Therefore, when we substitute back into (1), we get:

$$r_i' = \mathrm{Tr}\Big(U^\dagger \sigma_i U\, \rho\Big) = \mathrm{Tr}\left(\sum_j R_{ij}\sigma_j \rho\right) = \sum_j R_{ij}\,\mathrm{Tr}(\sigma_j \rho) = \sum_j R_{ij} r_j,$$

so $\vec{r}\,' = R\vec{r}$ for a real 3×3 matrix R.

To show orthogonality ($R^T R = \mathbb{1}$), evaluate $\mathrm{Tr}\big((U^\dagger \sigma_i U)(U^\dagger \sigma_k U)\big)$ in two ways. First, using the unitary property and the cyclic property of the trace:

$$\mathrm{Tr}\Big(U^\dagger \sigma_i U\, U^\dagger \sigma_k U\Big) = \mathrm{Tr}\Big(U^\dagger \sigma_i \sigma_k U\Big) = \mathrm{Tr}(\sigma_i \sigma_k) = 2\delta_{ik}$$

Second, using the newly derived expansion in terms of R:

$$\begin{aligned}\mathrm{Tr}\left(\sum_j R_{ij}\sigma_j \sum_\ell R_{k\ell}\sigma_\ell\right) &= \sum_{j,\ell} R_{ij}R_{k\ell}\,\mathrm{Tr}(\sigma_j\sigma_\ell) \\ &= \sum_{j,\ell} R_{ij}R_{k\ell}(2\delta_{j\ell}) \\ &= 2\sum_j R_{ij}R_{kj}\end{aligned}$$

Equating the two results gives $2\sum_j R_{ij}R_{kj} = 2\delta_{ik}$, which means $(RR^T)_{ik} = \delta_{ik}$. Hence $RR^T = \mathbb{1}$ (equivalently $R^T R = \mathbb{1}$).

Since R is a real orthogonal matrix, it preserves inner products and vector lengths:

$$\left\|\vec{r}\,'\right\|^2 = (R\vec{r}) \cdot (R\vec{r}) = \vec{r} \cdot (R^T R\vec{r}) = \vec{r} \cdot \vec{r} = \|\vec{r}\|^2$$

This rigid transformation that maps $\mathbb{R}^3 \to \mathbb{R}^3$ while keeping the origin fixed and preserving lengths corresponds to a rotation on the Bloch ball, preserving the purity of the state.

f) A qubit has Bloch vector $\vec{r} = (r_x, r_y, r_z)$. Show that the probability of obtaining outcome $|0\rangle$ when measuring in the computational basis is

$$p(0) = \frac{1 + r_z}{2}.$$

Compute $p(0)$ and $p(1)$ for the state ρ_1 from part (c). What are the probabilities if the measurement is instead performed in the $\{|+\rangle, |-\rangle\}$ basis?

Solution: We have

$$p(0) = \mathrm{Tr}(|0\rangle\langle 0|\,\rho).$$

Using $\rho = \frac{1}{2}(\mathbb{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$ and $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$$p(0) = \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}\right) = \frac{1}{2}(1 + r_z).$$

Hence $p(0) = \frac{1+r_z}{2}$ and $p(1) = 1 - p(0) = \frac{1-r_z}{2}$.

For ρ_1 we found $\vec{r} = (0, 0, 1/2)$, so

$$p(0) = \frac{1 + 1/2}{2} = \frac{3}{4}, \qquad p(1) = \frac{1}{4}.$$

For measurement in the $\{|+\rangle, |-\rangle\}$ basis,

$$p(+) = \mathrm{Tr}(|+\rangle\langle+|\,\rho) = \mathrm{Tr}\left(\frac{\mathbb{1} + \sigma_x}{2}\,\rho\right) = \frac{1}{2}\,\mathrm{Tr}(\rho) + \frac{1}{2}\,\mathrm{Tr}(\sigma_x\rho) = \frac{1}{2}\cdot 1 + \frac{1}{2}\,r_x = \frac{1 + r_x}{2}.$$

Similarly $p(-) = \frac{1-r_x}{2}$. For ρ_1, $r_x = 0$, so $p(+) = p(-) = \frac{1}{2}$.

Problem 2: Reduced Density Operators and the Partial Trace

a) Consider the two-qubit state $|\psi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$.

i) Compute the density matrix $\rho_{AB} = |\psi\rangle\langle\psi|$.

ii) Compute the reduced density operator $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$.

iii) For what values of α and β is ρ_A maximally mixed (i.e. $\rho_A = \mathbb{1}/2$)?

Solution:

i) The full density matrix is:

$$\rho_{AB} = |\alpha|^2\,|00\rangle\langle 00| + \alpha\beta^*\,|00\rangle\langle 11| + \alpha^*\beta\,|11\rangle\langle 00| + |\beta|^2\,|11\rangle\langle 11|.$$

ii) Taking the partial trace over B:

$$\begin{aligned}\rho_A &= |\alpha|^2\,|0\rangle\langle 0|\underbrace{\langle 0|0\rangle}_{1} + \alpha\beta^*\,|0\rangle\langle 1|\underbrace{\langle 1|0\rangle}_{0} + \alpha^*\beta\,|1\rangle\langle 0|\underbrace{\langle 0|1\rangle}_{0} + |\beta|^2\,|1\rangle\langle 1|\underbrace{\langle 1|1\rangle}_{1} \\ &= |\alpha|^2\,|0\rangle\langle 0| + |\beta|^2\,|1\rangle\langle 1| = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}.\end{aligned}$$

By symmetry of the state under swapping $A \leftrightarrow B$, we also have

$$\rho_B = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}.$$

iii) $\rho_A = \mathbb{1}/2$ holds iff $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$, i.e. $|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$. Writing $\alpha = e^{i\theta}/\sqrt{2}$ and $\beta = e^{i(\theta+\phi)}/\sqrt{2}$, and ignoring the global phase $e^{i\theta}$, we may write

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + e^{i\phi}\,|11\rangle\big).$$

This is maximally entangled for any ϕ. Moreover it is locally equivalent to $|\Phi^+\rangle$: applying the local phase gate $U_B = |0\rangle\langle 0| + e^{-i\phi}\,|1\rangle\langle 1|$ gives $(\mathbb{1} \otimes U_B)\,|\psi\rangle = |\Phi^+\rangle$ (and $\phi = \pi$ gives $|\Phi^-\rangle$).

b) Consider the maximally mixed state $\rho = \mathbb{1}/2$.

i) Show that for any orthonormal basis $\{|e_0\rangle, |e_1\rangle\}$, the measurement probabilities on ρ

are $p(0) = p(1) = \frac{1}{2}$.

ii) Show that ρ is invariant under any unitary evolution: $U\rho U^\dagger = \rho$.

iii) Using these results, explain why $\mathbb{1}/2$ contains no information and what is its Bloch vector? What does maximal entanglement imply about the information that can be extracted about the global state from either qubit alone?

Solution:

i) For any orthonormal basis $\{|e_0\rangle, |e_1\rangle\}$:

$$p(k) = \mathrm{Tr}\big(|e_k\rangle\langle e_k| \cdot \tfrac{\mathbb{1}}{2}\big) = \frac{1}{2}\,\mathrm{Tr}(|e_k\rangle\langle e_k|) = \frac{1}{2}.$$

ii) Using $UU^\dagger = \mathbb{1}$:

$$U\,\frac{\mathbb{1}}{2}\,U^\dagger = \frac{UU^\dagger}{2} = \frac{\mathbb{1}}{2}.$$

iii) Parts i) and ii) imply that $\mathbb{1}/2$ carries no information: all measurements in any basis give completely random outcomes, and no unitary gate can change this. The Bloch vector is $\vec{r} = \vec{0}$ (center of the Bloch ball, from Problem 1). From part (a), since the reduced density matrices of the maximally entangled Bell state $|\Phi^+\rangle$ are $\rho_A = \rho_B = \mathbb{1}/2$, neither subsystem carries any information about the global state. All information is stored in correlations between the two qubits and is invisible to local measurements.

c) The *GHZ state* is a three-qubit entangled state defined as:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big).$$

i) Compute the reduced density operator ρ_{AB} obtained by tracing out qubit C.

ii) Is ρ_{AB} a pure or mixed state? Is it entangled? If it is not entangled, explain in 1–2 sentences how it is possible that ρ_{AB} is not entangled while $|\mathrm{GHZ}\rangle$ is.

Solution:

i) The full density matrix is:

$$\rho_{ABC} = \frac{1}{2}\big(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\big).$$

Taking the partial trace over C (using $\langle 0|0\rangle = \langle 1|1\rangle = 1$, $\langle 0|1\rangle = 0$):

$$\begin{aligned}\rho_{AB} &= \frac{1}{2}\big(|00\rangle\langle 00|\langle 0|0\rangle + |00\rangle\langle 11|\langle 1|0\rangle + |11\rangle\langle 00|\langle 0|1\rangle + |11\rangle\langle 11|\langle 1|1\rangle\big) \\ &= \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big) = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.\end{aligned}$$

ii) We check purity:

$$\rho_{AB}^2 = \frac{1}{4}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big) \quad\Longrightarrow\quad \mathrm{Tr}\big(\rho_{AB}^2\big) = \frac{1}{4}\big(\mathrm{Tr}(|00\rangle\langle 00|) + \mathrm{Tr}(|11\rangle\langle 11|)\big) = \frac{1}{2} \neq 1,$$

so ρ_{AB} is mixed.

The state ρ_{AB} is not a product state, but it is separable since

$$\rho_{AB} = \frac{1}{2}\big(|0\rangle\langle 0| \otimes |0\rangle\langle 0|\big) + \frac{1}{2}\big(|1\rangle\langle 1| \otimes |1\rangle\langle 1|\big),$$

which is a convex combination of product states. Therefore ρ_{AB} is not entangled.

The GHZ state has genuine tripartite entanglement (e.g., it is entangled across the bipartition $AB|C$), but tracing out C discards the phase coherence between the $|000\rangle$ and $|111\rangle$ components, leaving only a classical mixture on AB, which is not entangled.

Problem 3: Schmidt Decomposition and Purification

This problem connects the Schmidt decomposition of bipartite pure states with reduced density operators and the purification of mixed states. Unless stated otherwise, all systems are qubits and we use the computational basis $\{|0\rangle, |1\rangle\}$.

a) General properties of the Schmidt decomposition and reduced states.

Let $|\psi\rangle_{AB}$ be a bipartite pure state with Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \lambda_i \, |i_A\rangle \, |i_B\rangle, \qquad \lambda_i > 0, \quad \sum_{i=1}^{r} \lambda_i^2 = 1,$$

and define the reduced states $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$.

i) Prove that $|\psi\rangle$ is a product state if and only if its Schmidt number is $r = 1$.

ii) Show that the Schmidt number is invariant under local unitaries, i.e. under $(U_A \otimes U_B)\,|\psi\rangle$.

iii) Using the Schmidt decomposition, find expressions for ρ_A and ρ_B. What are their nonzero eigenvalues? Deduce that

$$\mathrm{rank}(\rho_A) = \mathrm{rank}(\rho_B) = r.$$

iv) Explain in 2–4 sentences why for bipartite pure states, entanglement is reflected in the mixedness of reduced states.

v) Assume you have diagonalized ρ_A as

$$\rho_A = \sum_{i=1}^{r} p_i \, |i_A\rangle\langle i_A|, \qquad p_i > 0.$$

Define unnormalized states

$$|\widetilde{i}_B\rangle = (\langle i_A| \otimes \mathbb{1}_B)\,|\psi\rangle \in \mathcal{H}_B.$$

Show that $\| \, |\widetilde{i}_B\rangle\|^2 = p_i$ and that the normalized vectors

$$|i_B\rangle = \frac{1}{\sqrt{p_i}} \, |\widetilde{i}_B\rangle$$

form an orthonormal set in B. Conclude that

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \sqrt{p_i} \, |i_A\rangle \, |i_B\rangle$$

is a Schmidt decomposition.

This gives a systematic way of computing the Schmidt decomposition of any bipartite pure state: compute $\rho_A \mapsto$ diagonalize ρ_A to identify p_i and $|i_A\rangle \mapsto$ use the above method to find $|i_B\rangle \mapsto$ write down the Schmidt decomposition.

Solution:

i) ($\Leftarrow$): If $r = 1$, then $|\psi\rangle = \lambda_1 |1_A\rangle |1_B\rangle$ and normalization gives $\lambda_1 = 1$, so $|\psi\rangle$ is a product state.

($\Rightarrow$): If $|\psi\rangle = |a\rangle_A |b\rangle_B$ is a product state, then it clearly has Schmidt decomposition with exactly one nonzero Schmidt coefficient (equal to 1), so $r = 1$.

ii) If $|\psi\rangle = \sum_{i=1}^r \lambda_i |i_A\rangle |i_B\rangle$ then

$$(U_A \otimes U_B) |\psi\rangle = \sum_{i=1}^r \lambda_i (U_A |i_A\rangle) (U_B |i_B\rangle).$$

Since U_A and U_B are unitary, $\{U_A |i_A\rangle\}$ and $\{U_B |i_B\rangle\}$ are orthonormal sets, so this is a Schmidt decomposition with the same nonzero Schmidt coefficients λ_i. Hence the Schmidt number is invariant under local unitaries.

iii) From Schmidt form,

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \sum_{i=1}^r \lambda_i^2 |i_A\rangle\langle i_A|, \qquad \rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \sum_{i=1}^r \lambda_i^2 |i_B\rangle\langle i_B|.$$

Thus the nonzero eigenvalues of ρ_A and ρ_B are the same set $\{\lambda_i^2\}_{i=1}^r$. In particular, $\mathrm{rank}(\rho_A) = \mathrm{rank}(\rho_B) = r$ since the rank counts the number of nonzero eigenvalues, i.e. the number of nonzero Schmidt coefficients.

iv) For a bipartite pure state, the reduced states have eigenvalues λ_i^2, where λ_i are Schmidt coefficients. A product state has exactly one nonzero Schmidt coefficient, so the reduced state is pure (rank 1). An entangled pure state has at least two nonzero Schmidt coefficients, so the reduced states have rank ≥ 2 and are mixed. In fixed dimension, "more balanced" Schmidt coefficients (closer to uniform) correspond to more mixed reduced states, and (for pure states) to more entanglement across the bipartition.

v) First note that $\sum_{i=1}^r |i_A\rangle\langle i_A|$ is the projector onto the support of ρ_A. Since $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ has support contained in $\mathrm{span}\{|i_A\rangle\}_{i=1}^r$, the vector $|\psi\rangle$ has no component outside this support on system A. Therefore

$$|\psi\rangle = \Big(\sum_{i=1}^r |i_A\rangle\langle i_A| \otimes \mathbb{1}_B\Big) |\psi\rangle = \sum_{i=1}^r |i_A\rangle (\langle i_A| \otimes \mathbb{1}_B) |\psi\rangle = \sum_{i=1}^r |i_A\rangle |\widetilde{i}_B\rangle.$$

Now compute inner products:

$$\langle\widetilde{j}_B|\widetilde{i}_B\rangle = \langle\psi|\big(|j_A\rangle\langle i_A| \otimes \mathbb{1}_B\big) |\psi\rangle = \langle i_A| \rho_A |j_A\rangle = p_i \delta_{ij}.$$

Setting $i = j$ gives $\| |\widetilde{i}_B\rangle\|^2 = p_i$. For $i \neq j$ we get orthogonality. Therefore the normalized vectors $|i_B\rangle = |\widetilde{i}_B\rangle/\sqrt{p_i}$ form an orthonormal set, and

$$|\psi\rangle = \sum_{i=1}^r |i_A\rangle |\widetilde{i}_B\rangle = \sum_{i=1}^r \sqrt{p_i}\ |i_A\rangle |i_B\rangle$$

is a Schmidt decomposition (with Schmidt coefficients $\lambda_i = \sqrt{p_i}$).

b) Schmidt decomposition computation.

Consider the two-qubit state

$$|\psi_2\rangle_{AB} = \frac{\sqrt{3}+1}{4}\big(|00\rangle + |11\rangle\big) + \frac{\sqrt{3}-1}{4}\big(|01\rangle + |10\rangle\big).$$

i) Compute $\rho_A = \mathrm{Tr}_B(|\psi_2\rangle\langle\psi_2|)$ in the computational basis $\{|0\rangle, |1\rangle\}$ of A.

ii) Diagonalize ρ_A: find its eigenvalues and an orthonormal eigenbasis.

iii) Use your diagonalization to find the Schmidt coefficients and Schmidt bases, and write the Schmidt decomposition of $|\psi_2\rangle$.

Solution:

i) Let

$$a = \frac{\sqrt{3}+1}{4}, \qquad b = \frac{\sqrt{3}-1}{4},$$

so that $|\psi_2\rangle = a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle)$. Using

$$\rho_A = \sum_{y=0}^{1} (\mathbb{1}_A \otimes \langle y|)\, |\psi_2\rangle\langle\psi_2| (\mathbb{1}_A \otimes |y\rangle),$$

we compute the matrix elements in the basis $\{|0\rangle, |1\rangle\}$:

$$(\rho_A)_{00} = |a|^2 + |b|^2, \qquad (\rho_A)_{11} = |b|^2 + |a|^2,$$

$$(\rho_A)_{01} = ab + ba = 2ab, \qquad (\rho_A)_{10} = 2ab.$$

Now

$$a^2 + b^2 = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{16} = \frac{(4+2\sqrt{3}) + (4-2\sqrt{3})}{16} = \frac{1}{2},$$

and

$$2ab = 2 \cdot \frac{(\sqrt{3}+1)(\sqrt{3}-1)}{16} = 2 \cdot \frac{3-1}{16} = \frac{1}{4}.$$

Hence

$$\rho_A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

ii) The eigenvalues of $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$ are

$$p_1 = \frac{3}{4}, \qquad p_2 = \frac{1}{4},$$

with corresponding orthonormal eigenvectors

$$|e_1\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |e_2\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Thus

$$\rho_A = \frac{3}{4}\,|+\rangle\langle+| + \frac{1}{4}\,|-\rangle\langle-|.$$

iii) The Schmidt coefficients are $\lambda_1 = \sqrt{p_1} = \frac{\sqrt{3}}{2}$ and $\lambda_2 = \sqrt{p_2} = \frac{1}{2}$. To obtain the Schmidt basis on B, use

$$|f_k\rangle_B = \frac{1}{\sqrt{p_k}}(\langle e_k| \otimes \mathbb{1}_B)\,|\psi_2\rangle.$$

A short calculation gives

$$(\langle +| \otimes \mathbb{1})\,|\psi_2\rangle = \frac{\sqrt{3}}{2}\,|+\rangle, \qquad (\langle -| \otimes \mathbb{1})\,|\psi_2\rangle = \frac{1}{2}\,|-\rangle.$$

Dividing by $\sqrt{p_1} = \frac{\sqrt{3}}{2}$ and $\sqrt{p_2} = \frac{1}{2}$ yields

$$|f_1\rangle = |+\rangle, \qquad |f_2\rangle = |-\rangle.$$

Therefore a Schmidt decomposition is

$$|\psi_2\rangle_{AB} = \frac{\sqrt{3}}{2}\,|+\rangle_A\,|+\rangle_B + \frac{1}{2}\,|-\rangle_A\,|-\rangle_B,$$

and the Schmidt number is 2.

c) Purification as Schmidt decomposition "in reverse".

i) Construct the standard purification $|\Psi_{\text{std}}\rangle_{AR}$ of the mixed state ρ_A you found in part (b) onto an ancilla system R.

ii) Verify that $\text{Tr}_R(|\Psi_{\text{std}}\rangle\langle\Psi_{\text{std}}|) = \rho_A$.

iii) Identify $R \cong B$. Show that $|\Psi_{\text{std}}\rangle_{AB}$ is generally not equal to the original state $|\psi_2\rangle_{AB}$ from the previous subproblem. Find an explicit unitary U_B such that

$$|\psi_2\rangle_{AB} = (\mathbb{1}_A \otimes U_B)\,|\Psi_{\text{std}}\rangle_{AB}.$$

Solution:

i) From the previous subproblem,

$$\rho_A = \frac{3}{4}\,|+\rangle\langle +| + \frac{1}{4}\,|-\rangle\langle -|,$$

so we can take

$$p_1 = \frac{3}{4}, \quad |e_1\rangle = |+\rangle, \qquad p_2 = \frac{1}{4}, \quad |e_2\rangle = |-\rangle.$$

The standard purification is

$$|\Psi_{\text{std}}\rangle_{AR} = \sqrt{\frac{3}{4}}\,|+\rangle_A\,|0\rangle_R + \sqrt{\frac{1}{4}}\,|-\rangle_A\,|1\rangle_R = \frac{\sqrt{3}}{2}\,|+\rangle_A\,|0\rangle_R + \frac{1}{2}\,|-\rangle_A\,|1\rangle_R.$$

ii) Expand the projector:

$$|\Psi_{\text{std}}\rangle\langle\Psi_{\text{std}}| = \sum_{i,j\in\{1,2\}} \sqrt{p_i p_j}\;|e_i\rangle\langle e_j|_A \otimes |i\rangle\langle j|_R.$$

Using $\text{Tr}(|i\rangle\langle j|) = \delta_{ij}$,

$$\text{Tr}_R(|\Psi_{\text{std}}\rangle\langle\Psi_{\text{std}}|) = \frac{3}{4}\,|+\rangle\langle +| + \frac{1}{4}\,|-\rangle\langle -| = \rho_A.$$

iii) After identifying $R \cong B$ via the computational basis,

$$|\Psi_{\text{std}}\rangle_{AB} = \frac{\sqrt{3}}{2}\,|+\rangle_A\,|0\rangle_B + \frac{1}{2}\,|-\rangle_A\,|1\rangle_B.$$

But from the Schmidt decomposition found in the previous subproblem,

$$|\psi_2\rangle_{AB} = \frac{\sqrt{3}}{2}\,|+\rangle_A\,|+\rangle_B + \frac{1}{2}\,|-\rangle_A\,|-\rangle_B,$$

so indeed $|\Psi_{\text{std}}\rangle_{AB} \neq |\psi_2\rangle_{AB}$ (the B-side kets differ).

We want a unitary U_B satisfying

$$U_B\,|0\rangle = |+\rangle, \qquad U_B\,|1\rangle = |-\rangle.$$

The unique such unitary (up to an overall global phase) is the Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

since $H\,|0\rangle = |+\rangle$ and $H\,|1\rangle = |-\rangle$. Therefore

$$(\mathbb{1}_A \otimes H_B)\,|\Psi_{\text{std}}\rangle_{AB} = \frac{\sqrt{3}}{2}\,|+\rangle_A\,|+\rangle_B + \frac{1}{2}\,|-\rangle_A\,|-\rangle_B = |\psi_2\rangle_{AB}.$$

d) Consider a qubit state with Bloch vector $\vec{r} = (0, 0, r_z)$:

$$\rho = \frac{1}{2}\big(\mathbb{1} + r_z\sigma_z\big), \qquad -1 \le r_z \le 1.$$

- **i)** Diagonalize ρ and construct a purification $|\Psi(r_z)\rangle_{AR}$ using the eigenbasis of ρ.
- **ii)** Identify the Schmidt coefficients of $|\Psi(r_z)\rangle$ in terms of r_z, and state when the purification is maximally entangled and when it is a product state.

Solution:

i) Since $\rho = \frac{1}{2}(\mathbb{1} + r_z\sigma_z)$ is diagonal in the computational basis,

$$\rho = \frac{1 + r_z}{2}\,|0\rangle\langle 0| + \frac{1 - r_z}{2}\,|1\rangle\langle 1|,$$

where the eigenvalues are nonnegative precisely for $-1 \le r_z \le 1$. A standard purification is

$$|\Psi(r_z)\rangle_{AR} = \sqrt{\frac{1 + r_z}{2}}\;|0\rangle_A\,|0\rangle_R + \sqrt{\frac{1 - r_z}{2}}\;|1\rangle_A\,|1\rangle_R.$$

ii) This is already in Schmidt form across $A|R$. The Schmidt coefficients are

$$\lambda_1 = \sqrt{\frac{1 + r_z}{2}}, \qquad \lambda_2 = \sqrt{\frac{1 - r_z}{2}}.$$

For two qubits, the purification is maximally entangled when $\lambda_1 = \lambda_2$, i.e. $r_z = 0$ (the maximally mixed ρ), and it is a product state when one Schmidt coefficient is 0, i.e. when $r_z = \pm 1$ (the pure states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$).

Problem 4: Teleportation and Reduced Density Operators

In this task you revisit the quantum teleportation protocol, but now you will keep track of Bob's state using reduced density operators.

Let the input qubit (to be teleported) be

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Alice holds two qubits A_1 (the input) and A_2 (her half of the Bell pair), and Bob holds B (his half of the Bell pair). Initially,

$$|\Psi_0\rangle = |\psi\rangle_{A_1} \otimes \left|\Phi^+\right\rangle_{A_2B}, \qquad \left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Alice applies a CNOT with control A_1 and target A_2, followed by a Hadamard on A_1. The resulting three-qubit state (just before Alice measures) can be written as

$$|\Psi_2\rangle = \frac{1}{2}\Big(|00\rangle_{A_1A_2}\,|\psi\rangle_B + |01\rangle_{A_1A_2}\,\sigma_x\,|\psi\rangle_B + |10\rangle_{A_1A_2}\,\sigma_z\,|\psi\rangle_B + |11\rangle_{A_1A_2}\,\sigma_x\sigma_z\,|\psi\rangle_B\Big).$$

a) Compute Bob's reduced density operator before Alice measures,

$$\rho^B_{\text{pre}} = \mathrm{Tr}_{A_1A_2}\big(|\Psi_2\rangle\langle\Psi_2|\big).$$

Show that $\rho^B_{\text{pre}} = \mathbb{1}/2$, independent of α, β. Hint[3].

Solution: Let $\rho = |\psi\rangle\langle\psi|$. From the given form,

$$|\Psi_2\rangle = \frac{1}{2}\sum_m |m\rangle_{A_1A_2}\,|\phi_m\rangle_B$$

with

$$|\phi_{00}\rangle = |\psi\rangle,\ |\phi_{01}\rangle = \sigma_x\,|\psi\rangle,\ |\phi_{10}\rangle = \sigma_z\,|\psi\rangle,\ |\phi_{11}\rangle = \sigma_x\sigma_z\,|\psi\rangle.$$

Then

$$|\Psi_2\rangle\langle\Psi_2| = \frac{1}{4}\sum_{m,n} |m\rangle\langle n|_{A_1A_2} \otimes |\phi_m\rangle\langle\phi_n|_B.$$

Taking $\mathrm{Tr}_{A_1A_2}$ removes the $m \neq n$ terms:

$$\rho^B_{\text{pre}} = \mathrm{Tr}_{A_1A_2}(|\Psi_2\rangle\langle\Psi_2|) = \frac{1}{4}\sum_m |\phi_m\rangle\langle\phi_m| = \frac{1}{4}\Big(\rho + \sigma_x\rho\sigma_x + \sigma_z\rho\sigma_z + \sigma_x\sigma_z\rho\,\sigma_z\sigma_x\Big).$$

Write

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \equiv \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}, \quad a + b = 1.$$

A short calculation gives

$$\sigma_x\rho\sigma_x = \begin{pmatrix} b & c^* \\ c & a \end{pmatrix}, \qquad \sigma_z\rho\sigma_z = \begin{pmatrix} a & -c \\ -c^* & b \end{pmatrix}, \qquad \sigma_x\sigma_z\rho\,\sigma_z\sigma_x = \begin{pmatrix} b & -c^* \\ -c & a \end{pmatrix}.$$

Summing these four matrices:

$$\rho + \sigma_x\rho\sigma_x + \sigma_z\rho\sigma_z + \sigma_x\sigma_z\rho\,\sigma_z\sigma_x = \begin{pmatrix} 2(a+b) & 0 \\ 0 & 2(a+b) \end{pmatrix} = 2\mathbb{1}.$$

[3] Write $|\Psi_2\rangle = \frac{1}{2}\sum_{m\in\{00,01,10,11\}} |m\rangle_{A_1A_2}\,|\phi_m\rangle_B$.

Therefore

$$\rho^B_{\text{pre}} = \frac{1}{4} \cdot 2\mathbb{1} = \frac{\mathbb{1}}{2},$$

which is independent of α, β.

b) Alice measures (A_1, A_2) in the computational basis and gets a two-bit outcome $m \in \{00, 01, 10, 11\}$.

i) Show that $p(m) = \frac{1}{4}$ for all outcomes.

ii) For each m, write Bob's (normalized) post-measurement density operator ρ^B_m.

Solution: Let $P_m = |m\rangle\langle m|_{A_1A_2} \otimes \mathbb{1}_B$ be the measurement projector.

i) From the decomposition,

$$P_m |\Psi_2\rangle = \frac{1}{2} |m\rangle_{A_1A_2} |\phi_m\rangle_B,$$

so

$$p(m) = \|P_m |\Psi_2\rangle\|^2 = \frac{1}{4} \| |\phi_m\rangle\|^2 = \frac{1}{4},$$

since each $|\phi_m\rangle$ is a normalized qubit state.

ii) The (normalized) post-measurement global state is $|m\rangle |\phi_m\rangle$, so Bob's post-measurement state is $|\phi_m\rangle$. Thus

$$\rho^B_{(00)} = \rho, \qquad \rho^B_{(01)} = \sigma_x \rho \sigma_x, \qquad \rho^B_{(10)} = \sigma_z \rho \sigma_z, \qquad \rho^B_{(11)} = \sigma_x \sigma_z \rho \, (\sigma_x \sigma_z)^\dagger = \sigma_x \sigma_z \rho \, \sigma_z \sigma_x.$$

c) Bob does not know Alice's measurement outcome yet. Compute Bob's state after Alice's measurement but before the classical message arrives,

$$\rho^B_{\text{after}} = \sum_m p(m) \, \rho^B_m.$$

Show that $\rho^B_{\text{after}} = \rho^B_{\text{pre}} = \mathbb{1}/2$. What does this imply about what Bob can learn *locally* at this stage?

Solution: Using the result from the previous part with $p(m) = 1/4$,

$$\rho^B_{\text{after}} = \frac{1}{4}\Big(\rho + \sigma_x \rho \sigma_x + \sigma_z \rho \sigma_z + \sigma_x \sigma_z \rho \, \sigma_z \sigma_x\Big).$$

But this is exactly the same expression we obtained for ρ^B_{pre} in the first subproblem, and we already showed it equals $\mathbb{1}/2$. Hence

$$\rho^B_{\text{after}} = \rho^B_{\text{pre}} = \frac{\mathbb{1}}{2}.$$

Therefore, before receiving Alice's two classical bits, Bob's qubit is maximally mixed and contains no information about the identity of $|\psi\rangle$: every measurement outcome on Bob's side is completely random (and independent of α, β). In particular, Bob cannot tell whether Alice has already performed her measurement.

d) After receiving $m = m_1 m_2$, Bob applies the usual teleportation correction

$$U_m = \sigma_z^{m_1} \sigma_x^{m_2}.$$

Show (using density operators) that Bob then recovers the teleported state:

$$\rho^B_{\text{corr}} = |\psi\rangle\langle\psi|.$$

Solution: Conditioned on m, Bob's (pre-correction) state is ρ^B_m, where from the decomposition we may write

$$\rho^B_m = V_m\,\rho\,V_m^\dagger, \qquad V_{00} = \mathbb{1},\ V_{01} = \sigma_x,\ V_{10} = \sigma_z,\ V_{11} = \sigma_x\sigma_z.$$

The correction is $U_m = \sigma_z^{m_1}\sigma_x^{m_2}$, which satisfies $U_m V_m = \mathbb{1}$. Therefore the corrected state is

$$\rho^B_{\text{corr}} = U_m\,\rho^B_m\,U_m^\dagger = U_m(V_m\rho V_m^\dagger)U_m^\dagger = (U_mV_m)\,\rho\,(U_mV_m)^\dagger = \rho = |\psi\rangle\langle\psi|.$$

So Bob can recover the teleported state only after receiving the two classical bits m.

e) In 2–4 sentences, briefly summarize why the teleportation protocol does not allow Alice to signal to Bob faster than light. In particular, explain why Bob cannot learn anything about $|\psi\rangle$ (or even whether Alice has already measured) before the two classical bits m arrive.

Solution: Before the classical message arrives, Bob's reduced state is maximally mixed: $\rho^B_{\text{pre}} = \rho^B_{\text{after}} = \mathbb{1}/2$, independent of α, β and independent of Alice's measurement outcome. Hence every local measurement on B has the same outcome statistics regardless of what Alice does in her lab, so Bob cannot tell whether Alice has measured, nor can he extract any information about $|\psi\rangle$. Only after receiving m (a classical, light-speed-limited signal) can Bob apply the correction and recover $|\psi\rangle$.

f) General proof of no-signaling.

Let ρ^{AB} be an arbitrary bipartite state (not necessarily pure), where A is Alice's entire system and B is Bob's system, and let $\rho^B = \text{Tr}_A(\rho^{AB})$ be Bob's reduced state. Alice measures her system with general measurement operators $\{M^A_m\}$, satisfying completeness $\sum_m (M^A_m)^\dagger M^A_m = \mathbb{1}_A$. The measurement outcomes m have probabilities $p(m)$ and corresponding post-measurement states ρ^{AB}_m. Let $\rho^B_m = \text{Tr}_A(\rho^{AB}_m)$ be Bob's reduced state conditioned on outcome m. Show that Bob's reduced state, averaged over outcomes in the usual way, is unchanged by the measurement:

$$\sum_m p(m)\,\rho^B_m = \text{Tr}_A(\rho^{AB}) = \rho^B.$$

Solution: For outcomes with $p(m) = 0$ the corresponding term $p(m)\rho^B_m$ is zero, so we may assume $p(m) > 0$.

Using the definition of ρ^{AB}_m,

$$p(m)\,\rho^B_m = p(m)\ \text{Tr}_A(\rho^{AB}_m) = \text{Tr}_A\Big[p(m)\rho^{AB}_m\Big] = \text{Tr}_A\Big[(M^A_m \otimes \mathbb{1}_B)\,\rho^{AB}\,(M^A_m \otimes \mathbb{1}_B)^\dagger\Big].$$

Summing over m gives

$$\sum_m p(m)\,\rho^B_m = \sum_m \text{Tr}_A\Big[(M^A_m \otimes \mathbb{1}_B)\,\rho^{AB}\,(M^A_m \otimes \mathbb{1}_B)^\dagger\Big].$$

Using cyclicity on Alice's system inside the partial trace,

$$\mathrm{Tr}_A\Big[(M_m^A \otimes \mathbb{1}_B)\,\rho^{AB}\,(M_m^A \otimes \mathbb{1}_B)^\dagger\Big] = \mathrm{Tr}_A\Big[\rho^{AB}\,\big((M_m^A)^\dagger M_m^A \otimes \mathbb{1}_B\big)\Big].$$

Therefore

$$\begin{aligned}\sum_m p(m)\,\rho_m^B &= \mathrm{Tr}_A\Big[\rho^{AB}\,\Big(\sum_m (M_m^A)^\dagger M_m^A \otimes \mathbb{1}_B\Big)\Big] \\ &= \mathrm{Tr}_A\Big[\rho^{AB}\,(\mathbb{1}_A \otimes \mathbb{1}_B)\Big] \\ &= \mathrm{Tr}_A(\rho^{AB}) \\ &= \rho^B.\end{aligned}$$

Thus, even though Alice's measurement changes the *conditional* state ρ_m^B, the *unconditional* (averaged) state available to Bob is unchanged: no-signaling.