University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 8

This problem set will be discussed on **Mar. 25**.

Problem 1: Kraus Operators with mixed environment

Recall that the dynamics of a system S interacting with an environment E (initially in a pure state $|e_0\rangle_E$) via a joint unitary U can be written as $\mathcal{E}(\rho_S) = \sum_k M_k \rho_S M_k^\dagger$, where $M_k = \langle e_k|_E U |e_0\rangle_E$ are the Kraus operators satisfying $\sum_k M_k^\dagger M_k = I_S$.

Use the concept of purification to prove that the assumption of an initially pure environment can be made without loss of generality. Specifically, introduce a reference system R that purifies a mixed environment state ρ_E, form a larger pure environment $E' = E \otimes R$, and show that the reduced dynamics on S can still be written in the Kraus form $\mathcal{E}(\rho_S) = \sum_{j,k} \tilde{M}_{jk} \rho_S \tilde{M}_{jk}^\dagger$ and that the Kraus operators $\tilde{M}_{jk}$ still satisfy $\sum_{j,k} \tilde{M}_{jk}^\dagger \tilde{M}_{jk} = I_S$.

Hint[1].

[1] Since ρ_E is given in its eigenbasis, the Schmidt decomposition guarantees that its purification takes the single-index form $|\Psi\rangle_{ER} = \sum_i \sqrt{p_i} |e_i\rangle_E \otimes |r_i\rangle_R$, where $\{|r_i\rangle_R\}$ is an orthonormal set in R. Since the interaction only occurs between S and E, the joint unitary acts as $U_{SE} \otimes I_R$. Use the basis for the purified environment $\{|e_j\rangle_E |r_k\rangle_R\}$ to calculate the Kraus operators $\tilde{M}_{jk}$.

Problem 2: Naimark's Dilation Theorem

Naimark's dilation theorem states that any POVM on a system S can be realized as a projective measurement on a larger system $S \otimes A$. In this problem, you will construct this realization explicitly and explore its implementation using a simple quantum circuit.

a) Conceptual motivation. A standard projective measurement on a d-dimensional quantum system can have at most d outcomes. However, POVMs often describe measurements with more outcomes than the system's dimension (e.g., we explored Unambiguous State Discrimination which required a 3-outcome POVM on a single qubit).

Briefly explain why such a measurement cannot be implemented as a simple projective measurement directly on the system, and how Naimark's theorem conceptually bypasses this limitation.

b) Let $\{E_x\}$ be a POVM on S whose outcomes are labeled by n-bit strings $x \in \{0,1\}^n$. Define $M_x = \sqrt{E_x}$. Introduce an ancilla register A consisting of n qubits, and let $\{|x\rangle_A\}_{x\in\{0,1\}^n}$ denote its computational basis. Assume A is initialized in $|0\rangle_A^{\otimes n}$. Define an operator V acting on a pure state $|\psi\rangle_S$ by

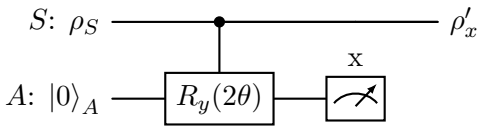
$$V|\psi\rangle_S = \sum_{x\in\{0,1\}^n} (M_x|\psi\rangle_S) \otimes |x\rangle_A.$$

i) Show that V is an isometry ($V^\dagger V = I_S$, meaning it preserves the inner product between any two states).

ii) The isometry V can always be extended to a unitary U on $S \otimes A$ satisfying $U(|\psi\rangle_S \otimes |0\rangle_A^{\otimes n}) = V|\psi\rangle_S$. Show that performing a projective measurement on A in the computational basis after applying U yields the POVM probabilities $p_x = \mathrm{Tr}(E_x\rho_S)$, and find the post-measurement state of S.

c) Consider a 2-outcome POVM with elements $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix}$ and $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$ where $0 < p < 1$.

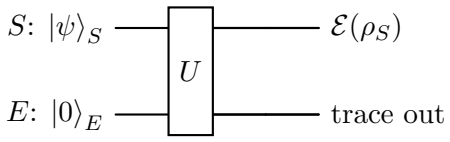
Verify that the following circuit with a single ancilla qubit implements the Naimark dilation for this POVM:



Find the angle θ such that a computational basis measurement on the ancilla reproduces exactly the POVM probability $p_1 = \mathrm{Tr}(E_1\rho_S)$.

Problem 3: Stinespring Dilation — Constructing the Unitary

In Problem 2, you saw how a POVM on a system can be realized as a projective measurement on a larger system. There is an analogous result for quantum *channels*: any channel $\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger$ can be realized by coupling to an environment, applying a joint unitary, and tracing out the environment:



But how do we explicitly *construct* this unitary from the Kraus operators? In this problem you will carry out this construction.

a) The general construction.

Given a quantum channel $\mathcal{E}(\rho) = \sum_{k=0}^{n-1} M_k \rho M_k^\dagger$ on a d-dimensional system S, we introduce an n-dimensional environment E with basis $\{|e_k\rangle\}$ initialized in $|e_0\rangle$, and define a unitary U on $S \otimes E$ by specifying its action:

$$U |\psi\rangle_S |e_0\rangle_E = \sum_k (M_k |\psi\rangle_S) \otimes |e_k\rangle_E.$$

i) Write U as a block matrix in the environment basis. That is, if $U = \sum_{j,k} U_{jk} \otimes |e_j\rangle\langle e_k|$ where each U_{jk} is a $d \times d$ block, show that the first block column is determined by the Kraus operators: $U_{k0} = M_k$.

ii) The remaining block columns of U are not fixed by the channel. Explain why they can always be chosen so that U is unitary, and what constraint the completeness relation $\sum_k M_k^\dagger M_k = I_S$ imposes on the first block column.

iii) Verify that $\mathrm{Tr}_E\big(U(\rho_S \otimes |e_0\rangle\langle e_0|_E)U^\dagger\big) = \sum_k M_k \rho_S M_k^\dagger$.

Hint[2].

b) Explicit example: the phase damping channel.

Consider the phase damping channel on a single qubit with Kraus operators

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}.$$

i) We want to construct a 4×4 unitary U on $S \otimes E$ (where E is a single qubit initialized in $|0\rangle_E$) that implements this channel via $U |\psi\rangle_S |0\rangle_E = M_0 |\psi\rangle_S |0\rangle_E + M_1 |\psi\rangle_S |1\rangle_E$. Write down the first two columns (the first block column) of U. Choose the remaining columns to make U unitary. Hint[3].

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[2] For ii), think of the columns of U as vectors in a dn-dimensional space. The first block column consists of d such columns. Unitarity requires all columns of U to form an orthonormal basis. Show that the completeness relation ensures that the d columns in the first block column are orthonormal.

[3] Writing U in 2×2 blocks, you have $U = \begin{pmatrix} M_0 & U_{01} \\ M_1 & U_{11} \end{pmatrix}$. The remaining blocks U_{01} and U_{11} must be chosen so that U is unitary. Use $U^\dagger U = I$ to find constraints. Be careful with the tensor product basis! Your matrix U is naturally constructed in the basis $E \otimes S$ due to the generic block definition from part (a). The standard matrix representation for systems, however, uses the $S \otimes E$ basis. You will need to account for this if you want to apply U directly as a standard 4×4 matrix.

ii) Let the initial state of the system be a generic pure state $|\psi\rangle_S = \alpha\,|0\rangle + \beta\,|1\rangle$, giving $\rho_S = |\psi\rangle\langle\psi|_S$. Verify that constructing $\mathcal{E}(\rho_S)$ using the Kraus operators yields the same 2×2 density matrix as when you apply your unitary U to the joint initial state $|\psi\rangle_S \otimes |0\rangle_E$ and trace out the environment.

c) In Problem 2 (Naimark dilation), you constructed an isometry from a POVM. In this problem (Stinespring dilation), you constructed an isometry from Kraus operators. Both rely on the completeness relation to guarantee the isometry property. Briefly explain why this structural similarity is not a coincidence: compare the roles of $\sum_m E_m = I$ (POVM) and $\sum_k M_k^\dagger M_k = I$ (Kraus) in enabling the construction.

Problem 4: Error Channels on the Bloch Sphere

The state of a single qubit can be represented by the density matrix:

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2} = \frac{1}{2} \begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$

where $\vec{\sigma} = (X, Y, Z)$ are the Pauli matrices and $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector with $|\vec{r}| \leq 1$. Quantum channels on a single qubit can thus be visualized as affine transformations of the Bloch sphere.

In this problem, we will explore how common error channels deform the Bloch sphere.

a) The bit flip channel. With probability p, the state of the qubit is flipped by applying the X gate, and with probability $(1 - p)$ the state is left alone. Write down the expression for the quantum channel $\mathcal{E}_{BF}(\rho)$ in terms of ρ and X. Then, calculate the new density matrix $\mathcal{E}_{BF}(\rho)$ in terms of the components of $\vec{r}$. Describe geometrically how the Bloch vector is transformed. Hint[4].

b) The amplitude damping channel. This channel models the process of energy loss (relaxation) to a zero-temperature environment. A qubit in the excited state $|1\rangle$ decays to the ground state $|0\rangle$ with probability γ, while a qubit already in $|0\rangle$ remains unchanged. The Kraus operators are:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Calculate the new density matrix $\mathcal{E}_{AD}(\rho)$ and extract the new components of the Bloch vector. Describe the geometric effect on the Bloch sphere, paying special attention to how it fundamentally differs from the bit flip channel.

[4] Remember the commutation and anti-commutation relations of the Pauli matrices, or simply multiply the matrices directly. For instance, $XXX = X$, $XYX = -Y$, and $XZX = -Z$.