University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 8

This problem set will be discussed on **Mar. 25**.

Problem 1: Kraus Operators with mixed environment

Recall that the dynamics of a system S interacting with an environment E (initially in a pure state $|e_0\rangle_E$) via a joint unitary U can be written as $\mathcal{E}(\rho_S) = \sum_k M_k \rho_S M_k^\dagger$, where $M_k = \langle e_k|_E U |e_0\rangle_E$ are the Kraus operators satisfying $\sum_k M_k^\dagger M_k = I_S$.

Use the concept of purification to prove that the assumption of an initially pure environment can be made without loss of generality. Specifically, introduce a reference system R that purifies a mixed environment state ρ_E, form a larger pure environment $E' = E \otimes R$, and show that the reduced dynamics on S can still be written in the Kraus form $\mathcal{E}(\rho_S) = \sum_{j,k} \tilde{M}_{jk} \rho_S \tilde{M}_{jk}^\dagger$ and that the Kraus operators $\tilde{M}_{jk}$ still satisfy $\sum_{j,k} \tilde{M}_{jk}^\dagger \tilde{M}_{jk} = I_S$.

Hint[1].

Solution: Let the initial state of the principal system be ρ_S. We introduce a reference system R whose sole purpose is to purify the environment E. Since $\rho_E = \sum_i p_i |e_i\rangle\langle e_i|_E$ is written in its eigenbasis, the Schmidt decomposition tells us that the purification takes the single-index form:

$$|\Psi\rangle_{ER} = \sum_i \sqrt{p_i} |e_i\rangle_E |r_i\rangle_R$$

where $\{|r_i\rangle_R\}$ is an orthonormal set in R with one basis vector per eigenvalue of ρ_E. One can verify that tracing out R recovers ρ_E: $\mathrm{Tr}_R(|\Psi\rangle\langle\Psi|_{ER}) = \sum_i p_i |e_i\rangle\langle e_i|_E = \rho_E$.

Now, we consider the interaction $U = U_{SE} \otimes I_R$ acting on the combined $S \otimes E \otimes R$ space. The new "super-environment" $E' = E \otimes R$ starts in the completely pure state $|\Psi\rangle_{ER}$.

To find the reduced dynamics on S, we trace out the combined pure environment E' using the joint basis $\{|e_j\rangle_E |r_k\rangle_R\}$. By the standard rules for a pure environment, the Kraus operators for this overall process are:

$$\begin{aligned}
\tilde{M}_{jk} &= (\langle e_j|_E \langle r_k|_R) U |\Psi\rangle_{ER} \\
&= (\langle e_j|_E \langle r_k|_R)(U_{SE} \otimes I_R) \left(\sum_i \sqrt{p_i} |e_i\rangle_E |r_i\rangle_R \right) \\
&= \sum_i \sqrt{p_i} \left(\langle e_j|_E U_{SE} |e_i\rangle_E \right) \langle r_k|r_i\rangle_R
\end{aligned}$$

Since $\langle r_k|r_i\rangle_R = \delta_{ki}$, the sum collapses to a single term:

$$\tilde{M}_{jk} = \sqrt{p_k} \langle e_j|_E U_{SE} |e_k\rangle_E$$

The enlarged environment $E' = E \otimes R$ starts in the pure state $|\Psi\rangle_{ER}$, so from the perspective of the system S, this is exactly the standard scenario of a unitary interaction with a pure

[1] Since ρ_E is given in its eigenbasis, the Schmidt decomposition guarantees that its purification takes the single-index form $|\Psi\rangle_{ER} = \sum_i \sqrt{p_i} |e_i\rangle_E \otimes |r_i\rangle_R$, where $\{|r_i\rangle_R\}$ is an orthonormal set in R. Since the interaction only occurs between S and E, the joint unitary acts as $U_{SE} \otimes I_R$. Use the basis for the purified environment $\{|e_j\rangle_E |r_k\rangle_R\}$ to calculate the Kraus operators $\tilde{M}_{jk}$.

environment. The standard operator-sum formalism therefore applies directly, and the reduced dynamics on S are:

$$\mathcal{E}(\rho_S) = \sum_{j,k} \tilde{M}_{jk} \rho_S \tilde{M}_{jk}^\dagger$$

Crucially, these are the same reduced dynamics that arise from the original physical interaction U_{SE} with the mixed environment ρ_E, since tracing out E' is equivalent to first tracing out R (which recovers ρ_E) and then tracing out E. This proves that any open system evolution — even one driven by a mixed environment — can always be rewritten as an interaction with a pure environment, so the assumption of a pure initial environment state is made without loss of generality.

To verify the completeness relation, we sum over both indices:

$$\begin{aligned}
\sum_{j,k} \tilde{M}_{jk}^\dagger \tilde{M}_{jk} &= \sum_{j,k} p_k \left(\langle e_k |_E U_{SE}^\dagger |e_j\rangle_E \right) \left(\langle e_j |_E U_{SE} |e_k\rangle_E \right) \\
&= \sum_k p_k \langle e_k |_E U_{SE}^\dagger \left(I_S \otimes \sum_j |e_j\rangle\langle e_j|_E \right) U_{SE} |e_k\rangle_E \\
&= \sum_k p_k \langle e_k |_E U_{SE}^\dagger (I_S \otimes I_E) U_{SE} |e_k\rangle_E \\
&= \sum_k p_k \langle e_k |_E (I_S \otimes I_E) |e_k\rangle_E \\
&= \sum_k p_k I_S \langle e_k | e_k \rangle_E = I_S \sum_k p_k = I_S.
\end{aligned}$$

The relation holds, completing the proof.

Problem 2: Naimark's Dilation Theorem

Naimark's dilation theorem states that any POVM on a system S can be realized as a projective measurement on a larger system $S \otimes A$. In this problem, you will construct this realization explicitly and explore its implementation using a simple quantum circuit.

a) Conceptual motivation. A standard projective measurement on a d-dimensional quantum system can have at most d outcomes. However, POVMs often describe measurements with more outcomes than the system's dimension (e.g., we explored Unambiguous State Discrimination which required a 3-outcome POVM on a single qubit).

Briefly explain why such a measurement cannot be implemented as a simple projective measurement directly on the system, and how Naimark's theorem conceptually bypasses this limitation.

Solution: The number of outcomes of a projective measurement corresponds to the maximum number of mutually orthogonal projectors that sum to the identity. In a d-dimensional space, there can be at most d nonzero mutually orthogonal projectors that sum to the identity. Therefore, any measurement with more than d outcomes cannot be projective on that space alone. Furthermore, projective measurements correspond to orthogonal measurement operators, while POVM elements can be non-orthogonal.

Naimark's theorem bypasses these limitations by introducing an ancilla, representing a physical interaction with an auxiliary quantum system (like a measurement apparatus). This creates a

joint Hilbert space $S \otimes A$ of a much larger dimension, providing enough "room" to construct a sufficient number of mutually orthogonal projectors. When we ignore the ancilla and restrict our view back to the original system, the statistics of these higher-dimensional projections perfectly reproduce the POVM.

b) Let $\{E_x\}$ be a POVM on S whose outcomes are labeled by n-bit strings $x \in \{0,1\}^n$. Define $M_x = \sqrt{E_x}$. Introduce an ancilla register A consisting of n qubits, and let $\{|x\rangle_A\}_{x\in\{0,1\}^n}$ denote its computational basis. Assume A is initialized in $|0\rangle_A^{\otimes n}$. Define an operator V acting on a pure state $|\psi\rangle_S$ by

$$V|\psi\rangle_S = \sum_{x\in\{0,1\}^n} (M_x|\psi\rangle_S) \otimes |x\rangle_A.$$

i) Show that V is an isometry ($V^\dagger V = I_S$, meaning it preserves the inner product between any two states).

ii) The isometry V can always be extended to a unitary U on $S \otimes A$ satisfying $U(|\psi\rangle_S \otimes |0\rangle_A^{\otimes n}) = V|\psi\rangle_S$. Show that performing a projective measurement on A in the computational basis after applying U yields the POVM probabilities $p_x = \mathrm{Tr}(E_x\rho_S)$, and find the post-measurement state of S.

Solution: i) Let $|\psi\rangle_S$ and $|\phi\rangle_S$ be arbitrary states. The inner product is:

$$\begin{aligned}(\langle\psi|_S V^\dagger)(V|\phi\rangle_S) &= \left(\sum_x \langle\psi|_S M_x^\dagger \otimes \langle x|_A\right)\left(\sum_y (M_y|\phi\rangle_S) \otimes |y\rangle_A\right)\\ &= \sum_{x,y} \langle\psi|_S M_x^\dagger M_y|\phi\rangle_S \langle x|y\rangle_A = \sum_x \langle\psi|_S M_x^\dagger M_x|\phi\rangle_S\\ &= \langle\psi|_S \left(\sum_x E_x\right)|\phi\rangle_S = \langle\psi|\phi\rangle_S.\end{aligned}$$

Since the inner product is preserved for all states, $V^\dagger V = I_S$.

ii) For a pure initial state $|\psi\rangle_S$, the joint state after applying U is:

$$|\Psi'\rangle_{SA} = U(|\psi\rangle_S \otimes |0\rangle_A^{\otimes n}) = V|\psi\rangle_S = \sum_x (M_x|\psi\rangle_S) \otimes |x\rangle_A.$$

The probability of outcome x from a projective measurement on A in the computational basis is:

$$\begin{aligned}p_x &= \left\|(I_S \otimes \langle x|_A)|\Psi'\rangle_{SA}\right\|^2 = \|M_x|\psi\rangle_S\|^2 = \langle\psi|_S M_x^\dagger M_x|\psi\rangle_S\\ &= \langle\psi|_S E_x|\psi\rangle_S = \mathrm{Tr}(E_x|\psi\rangle\langle\psi|_S).\end{aligned}$$

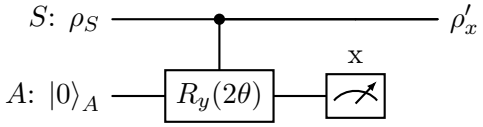
By linearity, this extends to mixed states: $p_x = \mathrm{Tr}(E_x\rho_S)$.

The post-measurement state of S, conditioned on outcome x, is:

$$\rho_x' = \frac{M_x\rho_S M_x^\dagger}{p_x}.$$

c) Consider a 2-outcome POVM with elements $E_0 = \begin{pmatrix}1 & 0\\ 0 & 1-p\end{pmatrix}$ and $E_1 = \begin{pmatrix}0 & 0\\ 0 & p\end{pmatrix}$ where $0 < p < 1$.

Verify that the following circuit with a single ancilla qubit implements the Naimark dilation for this POVM:



Find the angle θ such that a computational basis measurement on the ancilla reproduces exactly the POVM probability $p_1 = \mathrm{Tr}(E_1\rho_S)$.

Solution: Let's analyze the circuit. The initial state is $|\psi\rangle_S|0\rangle_A = (\alpha|0\rangle_S + \beta|1\rangle_S)|0\rangle_A$. The controlled-$R_y(2\theta)$ gate applies $R_y(2\theta)$ to the ancilla A only if the system S is in state $|1\rangle_S$.

The state after the controlled gate is:

$$\begin{aligned}|\Psi'\rangle_{SA} &= \alpha|0\rangle_S|0\rangle_A + \beta|1\rangle_S R_y(2\theta)|0\rangle_A \\ &= \alpha|0\rangle_S|0\rangle_A + \beta|1\rangle_S(\cos\theta|0\rangle_A + \sin\theta|1\rangle_A) \\ &= (\alpha|0\rangle_S + \beta\cos\theta|1\rangle_S)\otimes|0\rangle_A + (\beta\sin\theta|1\rangle_S)\otimes|1\rangle_A.\end{aligned}$$

A measurement of the ancilla in the computational basis yields outcome $x = 1$ with probability:

$$p'_1 = |\beta\sin\theta|^2 = |\beta|^2\sin^2\theta.$$

On the other hand, the probability p_1 predicted by the POVM element E_1 for the state $|\psi\rangle_S = \alpha|0\rangle + \beta|1\rangle$ is:

$$p_1 = \langle\psi|_S E_1|\psi\rangle_S = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & p\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = p|\beta|^2.$$

For the circuit measurement to reproduce the POVM exactly, we must have $p'_1 = p_1$ for all states, which requires $\sin^2\theta = p$, thus $\sin\theta = \sqrt{p}$.

Equivalently, the induced measurement operators on S are

$$K_0 = (I_S\otimes\langle 0|_A)\,U\,(I_S\otimes|0\rangle_A) = \begin{pmatrix}1 & 0\\ 0 & \cos\theta\end{pmatrix}, \qquad K_1 = (I_S\otimes\langle 1|_A)\,U\,(I_S\otimes|0\rangle_A) = \begin{pmatrix}0 & 0\\ 0 & \sin\theta\end{pmatrix}.$$

With $\sin^2\theta = p$ (and hence $\cos^2\theta = 1-p$), this gives

$$K_0 = \sqrt{E_0}, \qquad K_1 = \sqrt{E_1},$$

so the circuit reproduces the desired POVM as a Naimark dilation.

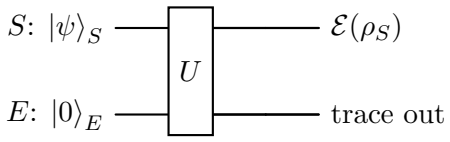
Let us also verify the other outcome $x = 0$. The probability is $p'_0 = |\alpha|^2 + |\beta\cos\theta|^2$. Using $\cos^2\theta = 1 - \sin^2\theta = 1 - p$, we get $p'_0 = |\alpha|^2 + (1-p)|\beta|^2$. This matches the POVM prediction exactly:

$$p_0 = \langle\psi|_S E_0|\psi\rangle_S = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1-p\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = |\alpha|^2 + (1-p)|\beta|^2.$$

Thus, this simple circuit and a choice of $\theta = \arcsin\left(\sqrt{p}\right)$ perfectly implements the Naimark dilation for this POVM.

Problem 3: Stinespring Dilation — Constructing the Unitary

In Problem 2, you saw how a POVM on a system can be realized as a projective measurement on a larger system. There is an analogous result for quantum *channels*: any channel $\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger$ can be realized by coupling to an environment, applying a joint unitary, and tracing out the environment:



But how do we explicitly *construct* this unitary from the Kraus operators? In this problem you will carry out this construction.

a) The general construction.

Given a quantum channel $\mathcal{E}(\rho) = \sum_{k=0}^{n-1} M_k \rho M_k^\dagger$ on a d-dimensional system S, we introduce an n-dimensional environment E with basis $\{|e_k\rangle\}$ initialized in $|e_0\rangle$, and define a unitary U on $S \otimes E$ by specifying its action:

$$U |\psi\rangle_S |e_0\rangle_E = \sum_k (M_k |\psi\rangle_S) \otimes |e_k\rangle_E.$$

i) Write U as a block matrix in the environment basis. That is, if $U = \sum_{j,k} U_{jk} \otimes |e_j\rangle\langle e_k|$ where each U_{jk} is a $d \times d$ block, show that the first block column is determined by the Kraus operators: $U_{k0} = M_k$.

ii) The remaining block columns of U are not fixed by the channel. Explain why they can always be chosen so that U is unitary, and what constraint the completeness relation $\sum_k M_k^\dagger M_k = I_S$ imposes on the first block column.

iii) Verify that $\mathrm{Tr}_E\big(U(\rho_S \otimes |e_0\rangle\langle e_0|_E)U^\dagger\big) = \sum_k M_k \rho_S M_k^\dagger$.

Hint[2].

Solution:

i) The block matrix representation of U in the basis $\{|e_k\rangle_E\}$ is:

$$U = \begin{pmatrix} [U_{00}] & [U_{01}] & \cdots \\ [U_{10}] & [U_{11}] & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where U_{jk} is the $d \times d$ block satisfying $U_{jk} = \langle e_j|_E U |e_k\rangle_E$. From the defining relation $U |\psi\rangle_S |e_0\rangle_E = \sum_k (M_k |\psi\rangle_S) |e_k\rangle_E$, we read off $U_{k0} = \langle e_k|_E U |e_0\rangle_E = M_k$ (by definition). So the first block column is:

$$U = \begin{pmatrix} [M_0] & \cdot & \cdots \\ [M_1] & \cdot & \cdots \\ [M_2] & \cdot & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

[2] For ii), think of the columns of U as vectors in a dn-dimensional space. The first block column consists of d such columns. Unitarity requires all columns of U to form an orthonormal basis. Show that the completeness relation ensures that the d columns in the first block column are orthonormal.

ii) The matrix U acts on the joint space $S \otimes E$, which has dimension $d \cdot n = dn$. For U to be unitary, its dn columns must form an orthonormal basis for this space. This requires $U^\dagger U = I_{dn}$.

By the definition of block matrix multiplication, the top-left $d \times d$ block of $U^\dagger U$ evaluates the inner products between the first d columns of U. Since the first block-column of U is composed of the n vertically stacked blocks $M_0, M_1, \ldots, M_{n-1}$, we evaluate this block product as:

$$(U^\dagger U)_{00} = \sum_{k=0}^{n-1} U_{k0}^\dagger U_{k0} = \sum_{k=0}^{n-1} M_k^\dagger M_k$$

Because this sum spans all n blocks in the block-column, it accounts for the entire dn height of these vectors. For $U^\dagger U = I_{dn}$ to hold, this top-left block must equal the $d \times d$ identity matrix I_S. Therefore, the completeness relation $\sum_k M_k^\dagger M_k = I_S$ is precisely the mathematical requirement that the first d columns of U form a set of d mutually orthonormal vectors in $\mathbb{C}^{dn}$.

Finally, any set of orthonormal vectors in a finite-dimensional space can be extended to a full orthonormal basis (for example, by using the Gram-Schmidt orthogonalization procedure). The completeness relation thus guarantees that we can always construct the remaining $dn - d$ orthonormal columns needed to populate the rest of the matrix U, making it unitary.

iii) By definition, the action of U on an initial pure state $|\psi\rangle_S \otimes |e_0\rangle_E$ yields a superposition over the environment basis states:

$$U(|\psi\rangle_S \otimes |e_0\rangle_E) = \sum_j (M_j |\psi\rangle_S) \otimes |e_j\rangle_E$$

Assuming $\rho_S = |\psi\rangle\langle\psi|_S$ (which extends to mixed states by linearity), we form the joint density matrix after the unitary by taking the outer product:

$$\begin{aligned} U(\rho_S \otimes |e_0\rangle\langle e_0|_E)U^\dagger &= \left(\sum_j (M_j |\psi\rangle_S) \otimes |e_j\rangle_E\right)\left(\sum_k (\langle\psi|_S M_k^\dagger) \otimes \langle e_k|_E\right) \\ &= \sum_{j,k} (M_j \rho_S M_k^\dagger) \otimes |e_j\rangle\langle e_k|_E. \end{aligned}$$

Tracing over E:

$$\mathrm{Tr}_E\big(U(\rho_S \otimes |e_0\rangle\langle e_0|)U^\dagger\big) = \sum_{j,k} M_j \rho_S M_k^\dagger \cdot \underbrace{\mathrm{Tr}(|e_j\rangle\langle e_k|)}_{\delta_{jk}} = \sum_k M_k \rho_S M_k^\dagger = \mathcal{E}(\rho_S). \qquad \checkmark$$

b) Explicit example: the phase damping channel.

Consider the phase damping channel on a single qubit with Kraus operators

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}.$$

i) We want to construct a 4×4 unitary U on $S \otimes E$ (where E is a single qubit initialized in $|0\rangle_E$) that implements this channel via $U|\psi\rangle_S |0\rangle_E = M_0 |\psi\rangle_S |0\rangle_E + M_1 |\psi\rangle_S |1\rangle_E$. Write down the first two columns (the first block column) of U. Choose the remaining columns to make U unitary. Hint[3].

[3] Writing U in 2×2 blocks, you have $U = \begin{pmatrix} M_0 & U_{01} \\ M_1 & U_{11} \end{pmatrix}$. The remaining blocks U_{01} and U_{11} must be chosen so

ii) Let the initial state of the system be a generic pure state $|\psi\rangle_S = \alpha|0\rangle + \beta|1\rangle$, giving $\rho_S = |\psi\rangle\langle\psi|_S$. Verify that constructing $\mathcal{E}(\rho_S)$ using the Kraus operators yields the same 2×2 density matrix as when you apply your unitary U to the joint initial state $|\psi\rangle_S \otimes |0\rangle_E$ and trace out the environment.

Solution:

i) From the given action, the first block column of U is fixed: $U_{00} = M_0$, $U_{10} = M_1$. We need to find U_{01} and U_{11} such that U is unitary.

Since U is unitary, the two block columns have to be orthogonal and normalized. Writing out the block matrix U as

$$U = \begin{pmatrix} M_0 & U_{01} \\ M_1 & U_{11} \end{pmatrix},$$

we get that $M_0^\dagger U_{01} + M_1^\dagger U_{11} = 0$ from the orthogonality condition, and $U_{01}^\dagger U_{01} + U_{11}^\dagger U_{11} = I$ from the normalization condition.

Since M_0 and M_1 are both diagonal and real, we can search for a solution where U_{01} and U_{11} are also diagonal. Let $U_{01} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $U_{11} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$. The orthogonality condition $M_0^\dagger U_{01} + M_1^\dagger U_{11} = 0$ becomes:

$$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matching the matrix elements, we get two equations:

$$1 \cdot a + 0 \cdot c = 0 \implies a = 0$$
$$\sqrt{1-p} \cdot b + \sqrt{p} \cdot d = 0 \implies d = -\sqrt{\frac{1-p}{p}}b$$

The normalization condition $U_{01}^\dagger U_{01} + U_{11}^\dagger U_{11} = I$ gives two more equations for the diagonal elements:

$$|a|^2 + |c|^2 = 1 \implies 0 + |c|^2 = 1 \implies |c| = 1 \quad \text{(we can choose } c = 1\text{)}$$
$$|b|^2 + |d|^2 = 1 \implies |b|^2 + \frac{1-p}{p}|b|^2 = 1 \implies \frac{1}{p}|b|^2 = 1 \implies |b|^2 = p \quad \text{(we choose } b = \sqrt{p}\text{)}$$

Substituting $b = \sqrt{p}$ back into our expression for d, we find $d = -\sqrt{1-p}$. Thus, our successfully derived blocks are:

$$U_{01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}, \qquad U_{11} = \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{1-p} \end{pmatrix}.$$

These blocks gives us the full 4×4 matrix U in the generic block-basis of $E \otimes S$:

$$U_{E\otimes S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & \sqrt{p} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{p} & 0 & -\sqrt{1-p} \end{pmatrix}.$$

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that U is unitary. Use $U^\dagger U = I$ to find constraints. Be careful with the tensor product basis! Your matrix U is naturally constructed in the basis $E \otimes S$ due to the generic block definition from part (a). The standard matrix representation for systems, however, uses the $S \otimes E$ basis. You will need to account for this if you want to apply U directly as a standard 4×4 matrix.

If we reorder the tensor factors to write the full 4×4 matrix U in the standard computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of $S \otimes E$, the top-left elements of our blocks group together, as do the bottom-right elements. This reshuffles the matrix into a block-diagonal form:

$$U_{S\otimes E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & \sqrt{p} \\ 0 & 0 & \sqrt{p} & -\sqrt{1-p} \end{pmatrix}.$$

Note the clear block-diagonal structure of this matrix: the top-left 2×2 block corresponds to the system being in the $|0\rangle_S$ subspace, and it is simply the identity matrix. This means if the system is in $|0\rangle_S$, there is no interaction: the environment is completely unaffected (it stays in $|0\rangle_E$). The bottom-right 2×2 block corresponds to the $|1\rangle_S$ subspace, where the environment undergoes a rotation. This state-dependent rotation creates entanglement between the system and the environment. In other words, U acts exactly like a Controlled-Unitary operation where the system S is the control qubit! This matches the physical intuition that phase damping only affects the relative phases between $|0\rangle$ and $|1\rangle$ by entangling the system's phase information with the environment, without changing classical populations.

ii) To verify the output, let the system start in $|\psi\rangle_S = \alpha|0\rangle + \beta|1\rangle$, which gives the density matrix $\rho_S = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$. First, calculating the action of the channel directly from the Kraus operators M_0 and M_1:

$$\begin{aligned} \mathcal{E}(\rho_S) &= M_0\rho_S M_0^\dagger + M_1\rho_S M_1^\dagger \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \\ &= \begin{pmatrix} |\alpha|^2 & \sqrt{1-p}\,\alpha\beta^* \\ \sqrt{1-p}\,\alpha^*\beta & (1-p)|\beta|^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & p\,|\beta|^2 \end{pmatrix} \\ &= \begin{pmatrix} |\alpha|^2 & \sqrt{1-p}\,\alpha\beta^* \\ \sqrt{1-p}\,\alpha^*\beta & |\beta|^2 \end{pmatrix}. \end{aligned}$$

Second, we verify we get the same result by applying U and tracing out the environment. The initial joint pure state in the $S \otimes E$ basis is $|\Psi_{\text{in}}\rangle = |\psi\rangle_S \otimes |0\rangle_E = \alpha|00\rangle + \beta|10\rangle$. In column vector form:

$$|\Psi_{\text{in}}\rangle = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}.$$

Applying the unitary $U_{S\otimes E}$, the final joint state $|\Psi_{\text{out}}\rangle = U_{S\otimes E}|\Psi_{\text{in}}\rangle$ is:

$$\begin{aligned} |\Psi_{\text{out}}\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & \sqrt{p} \\ 0 & 0 & \sqrt{p} & -\sqrt{1-p} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \beta\sqrt{1-p} \\ \beta\sqrt{p} \end{pmatrix} \\ &= \alpha|0\rangle_S|0\rangle_E + \beta\sqrt{1-p}\,|1\rangle_S|0\rangle_E + \beta\sqrt{p}\,|1\rangle_S|1\rangle_E. \end{aligned}$$

Finally, we compute the partial trace over E to find the reduced density matrix of the system, $\rho'_S = \text{Tr}_E(|\Psi_{\text{out}}\rangle\langle\Psi_{\text{out}}|)$. This is done by collecting the system components paired with the

orthogonal environment states $|0\rangle_E$ and $|1\rangle_E$:

$$\begin{aligned}\rho'_S &= \left(\alpha|0\rangle_S + \beta\sqrt{1-p}\,|1\rangle_S\right)\left(\alpha^*\langle 0|_S + \beta^*\sqrt{1-p}\langle 1|_S\right) + \left(\beta\sqrt{p}\,|1\rangle_S\right)\left(\beta^*\sqrt{p}\langle 1|_S\right) \\ &= \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\sqrt{1-p} \\ \alpha^*\beta\sqrt{1-p} & |\beta|^2(1-p)\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & |\beta|^2 p\end{pmatrix} \\ &= \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\sqrt{1-p} \\ \alpha^*\beta\sqrt{1-p} & |\beta|^2\end{pmatrix}. \qquad \checkmark\end{aligned}$$

This identically matches the calculation using the Kraus operators. The diagonal entries (populations) are unchanged, while the off-diagonal entries (coherences) are damped by $\sqrt{1-p}$. This is why it is called *phase* damping.

c) In Problem 2 (Naimark dilation), you constructed an isometry from a POVM. In this problem (Stinespring dilation), you constructed an isometry from Kraus operators. Both rely on the completeness relation to guarantee the isometry property. Briefly explain why this structural similarity is not a coincidence: compare the roles of $\sum_m E_m = I$ (POVM) and $\sum_k M_k^\dagger M_k = I$ (Kraus) in enabling the construction.

Solution: Both constructions define an isometry $V|\psi\rangle = \sum_k (A_k|\psi\rangle)|k\rangle$ into a larger Hilbert space. For this to be an isometry, we need $V^\dagger V = \sum_k A_k^\dagger A_k = I$. In the POVM case, $A_m = M'_m = \sqrt{E_m}$, and we use $\sum_m {M'_m}^\dagger M'_m = \sum_m E_m = I$. In the Kraus case, $A_k = M_k$, and we use $\sum_k M_k^\dagger M_k = I$. Both conditions express the correct normalization: in the POVM case they ensure that the outcome probabilities sum to 1, and in the Kraus case they ensure trace preservation of the channel. This is exactly what allows the operator to be embedded as part of a unitary on the larger space. The deep connection is that POVMs describe the *measurement statistics* of quantum operations, while Kraus operators describe the *state transformation* — both are aspects of the same underlying framework of quantum operations.

Problem 4: Error Channels on the Bloch Sphere

The state of a single qubit can be represented by the density matrix:

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z\end{pmatrix}$$

where $\vec{\sigma} = (X, Y, Z)$ are the Pauli matrices and $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector with $|\vec{r}| \leq 1$. Quantum channels on a single qubit can thus be visualized as affine transformations of the Bloch sphere.

In this problem, we will explore how common error channels deform the Bloch sphere.

a) The bit flip channel. With probability p, the state of the qubit is flipped by applying the X gate, and with probability $(1-p)$ the state is left alone. Write down the expression for the quantum channel $\mathcal{E}_{BF}(\rho)$ in terms of ρ and X. Then, calculate the new density matrix $\mathcal{E}_{BF}(\rho)$ in terms of the components of $\vec{r}$. Describe geometrically how the Bloch vector is transformed. Hint[4].

[4] Remember the commutation and anti-commutation relations of the Pauli matrices, or simply multiply the matrices directly. For instance, $XXX = X$, $XYX = -Y$, and $XZX = -Z$.

Solution: The mathematical expression for the channel is:

$$\mathcal{E}_{BF}(\rho) = (1-p)\rho + pX\rho X$$

First, we evaluate the term $X\rho X$:

$$X\rho X = X\left(\frac{I + r_x X + r_y Y + r_z Z}{2}\right)X = \frac{I + r_x XXX + r_y XYX + r_z XZX}{2}$$

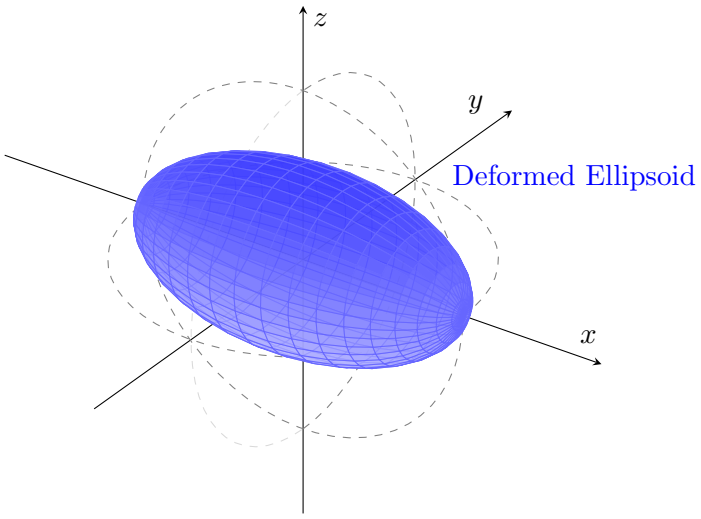
Since Pauli matrices anti-commute ($XY = -YX$, $XZ = -ZX$), we have $XYX = -YXX = -Y$ and $XZX = -ZXX = -Z$. Since $X^2 = I$, we get $XXX = X$.

$$X\rho X = \frac{I + r_x X - r_y Y - r_z Z}{2}$$

Now, we apply the channel:

$$\begin{aligned}\mathcal{E}_{BF}(\rho) &= (1-p)\left(\frac{I + r_x X + r_y Y + r_z Z}{2}\right) + p\left(\frac{I + r_x X - r_y Y - r_z Z}{2}\right) \\ &= \frac{I + r_x X + (1-2p) r_y Y + (1-2p) r_z Z}{2}\end{aligned}$$

The new Bloch vector is $\vec{r}_{BF} = (r_x, (1-2p)r_y, (1-2p)r_z)$. Geometrically, the x-component is unaffected, while the y and z components are multiplied by $(1-2p)$. Thus for $0 \leq p \leq \frac{1}{2}$ the Bloch sphere is contracted into an ellipsoid along the x-axis, while for $p > \frac{1}{2}$ there is also a reflection in the y- and z-directions.



b) The amplitude damping channel. This channel models the process of energy loss (relaxation) to a zero-temperature environment. A qubit in the excited state $|1\rangle$ decays to the ground state $|0\rangle$ with probability γ, while a qubit already in $|0\rangle$ remains unchanged. The Kraus operators are:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Calculate the new density matrix $\mathcal{E}_{AD}(\rho)$ and extract the new components of the Bloch vector. Describe the geometric effect on the Bloch sphere, paying special attention to how it fundamentally

differs from the bit flip channel.

Solution: First, we apply the channel components to the arbitrary density matrix $\rho = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}$:

$$\begin{aligned} M_0 \rho M_0^\dagger &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1+r_z & \sqrt{1-\gamma}(r_x - ir_y) \\ \sqrt{1-\gamma}(r_x + ir_y) & (1-\gamma)(1-r_z) \end{pmatrix} \end{aligned}$$

And for the second Kraus operator:

$$\begin{aligned} M_1 \rho M_1^\dagger &= \frac{1}{2}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sqrt{\gamma}(r_x + ir_y) & \sqrt{\gamma}(1-r_z) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \gamma(1-r_z) & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Summing these together to find $\mathcal{E}_{AD}(\rho)$:

$$\begin{aligned} \mathcal{E}_{AD}(\rho) &= M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger \\ &= \frac{1}{2}\begin{pmatrix} 1 + r_z + \gamma - \gamma r_z & \sqrt{1-\gamma}(r_x - ir_y) \\ \sqrt{1-\gamma}(r_x + ir_y) & (1-\gamma)(1-r_z) \end{pmatrix} \end{aligned}$$

To extract the new Bloch vector components r'_x, r'_y, r'_z, we match this result to the general form of $\rho = \frac{1}{2}\begin{pmatrix} 1+r'_z & r'_x - ir'_y \\ r'_x + ir'_y & 1-r'_z \end{pmatrix}$. The off-diagonal elements immediately give:

$$r'_x = \sqrt{1-\gamma} r_x, \qquad r'_y = \sqrt{1-\gamma} r_y$$

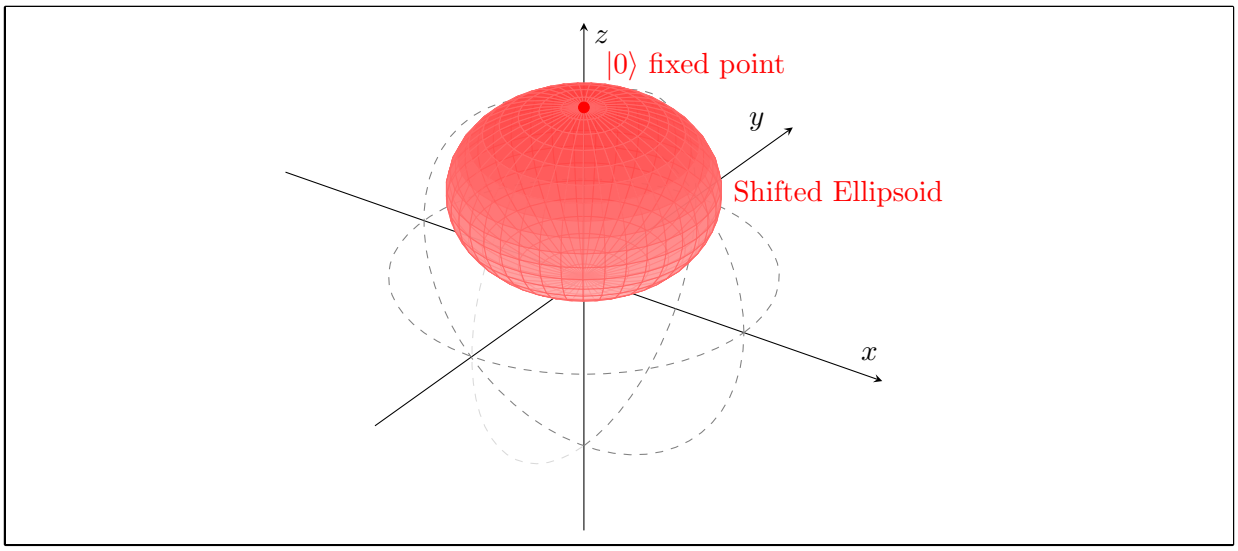
From the bottom-right element, we have $1 - r'_z = (1-\gamma)(1-r_z)$, which simplifies to:

$$r'_z = 1 - (1-\gamma) + r_z(1-\gamma) = \gamma + (1-\gamma) r_z$$

(A quick check of the top-left element confirms $1 + r'_z = 1 + \gamma + (1-\gamma) r_z$, cleanly matching).

The new Bloch vector is $\vec{r}_{AD} = (\sqrt{1-\gamma} r_x, \sqrt{1-\gamma} r_y, (1-\gamma) r_z + \gamma)$.

Geometrically, this differs from the bit flip channel because it is NOT merely a concentric contraction. The x and y components shrink by $\sqrt{1-\gamma}$, and the z component shrinks by $(1-\gamma)$, but there is also a constant positive shift of $+\gamma$ along the z-axis! This means the entire shrunken ellipsoid is translated upwards towards the North pole. Because amplitude damping is a non-unital channel (it does not leave the maximally mixed state $I/2$ invariant), the center of the sphere is displaced. However, the North pole itself ($|0\rangle$ with $r_z = 1$) remains perfectly fixed since $(1-\gamma)(1) + \gamma = 1$.


z
|0⟩ fixed point
y
Shifted Ellipsoid
x