University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 9

This problem set will be discussed on **Apr. 15**.

Problem 1: Trace Distance

For two quantum states ρ and σ, the trace distance is defined by

$$D(\rho, \sigma) = \frac{1}{2} \operatorname{Tr} |\rho - \sigma|.$$

a) Warm-up computation.

Consider the two single-qubit states

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \qquad \sigma = |+\rangle\langle +| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Calculate the trace distance $D(\rho, \sigma)$.

b) Maximum projector.

In lecture you proved that for any two states ρ, σ,

$$D(\rho, \sigma) = \max_{0 \leq E \leq I} \operatorname{Tr}(E(\rho - \sigma)),$$

where the maximum is over all POVM elements E. Let $|e_+\rangle$ be the eigenvector of $\rho - \sigma$ corresponding to the positive eigenvalue, and let $P_+ = |e_+\rangle\langle e_+|$.

Verify that

$$\operatorname{Tr}(P_+(\rho - \sigma)) = D(\rho, \sigma).$$

Problem 2: Fidelity and Uhlmann's theorem

The *fidelity* of density operators ρ and σ is

$$F(\rho,\sigma) \equiv \operatorname{Tr}\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}.$$

It is not a metric, but it measures similarity: $F(\rho,\sigma) = 1$ iff $\rho = \sigma$, and $F(\rho,\sigma) = 0$ iff the supports of ρ and σ are orthogonal.

In this problem you prove basic properties of fidelity and derive Uhlmann's theorem.

a) Special cases.

i) Let $\rho_\psi = |\psi\rangle\langle\psi|$ be a pure state. Show that for any density operator ρ,

$$F(\rho_\psi,\rho) = \sqrt{\langle\psi|\rho\,|\psi\rangle}.$$

ii) Let

$$\rho = \sum_i r_i\,|i\rangle\langle i|, \qquad \sigma = \sum_i s_i\,|i\rangle\langle i|$$

be commuting density operators. Show that

$$F(\rho,\sigma) = \sum_i \sqrt{r_i s_i}.$$

iii) Compute the fidelity between

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \qquad \sigma = \frac{1}{2}I.$$

b) Key lemma.

Show that for any operator A,

$$\max_U |\operatorname{Tr}(UA)| = \operatorname{Tr}|A|,$$

where $|A| = \sqrt{A^\dagger A}$ and the maximum is over all unitaries U. Hint[1].

c) Uhlmann's theorem.

Let ρ and σ be density operators on a system Q. Uhlmann's theorem states that

$$F(\rho,\sigma) = \max_{|\psi\rangle,|\phi\rangle} |\langle\psi|\phi\rangle|,$$

where the maximum is over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ onto $Q\otimes R$, with $R \cong Q$.

Prove this in the following steps.

i) Fix an orthonormal basis $\{|i\rangle\}$ of Q, use the corresponding basis on R, and define

$$|\Omega\rangle_{QR} = \sum_i |i\rangle_Q\,|i\rangle_R, \qquad |\psi_0\rangle_{QR} = (\sqrt{\rho}\otimes I_R)\,|\Omega\rangle, \qquad |\phi_0\rangle_{QR} = (\sqrt{\sigma}\otimes I_R)\,|\Omega\rangle.$$

Show that $|\psi_0\rangle$ and $|\phi_0\rangle$ are purifications of ρ and σ. Conclude that every purification of ρ onto $Q\otimes R$ has the form $(I_Q\otimes U_R)\,|\psi_0\rangle$ for some unitary U_R on R, and similarly for σ.

[1] Use the polar decomposition $A = V|A|$ for unitary V, and diagonalize $|A|$.

ii) Using

$$\langle\Omega|(A\otimes B)\,|\Omega\rangle = \mathrm{Tr}\big(A^T B\big),$$

where T denotes transpose in the chosen basis, show that

$$\langle\psi_0|(I_Q\otimes W_R)\,|\phi_0\rangle = \mathrm{Tr}\left(W^T\sqrt{\rho}\sqrt{\sigma}\right)$$

for any operator W_R on R.

iii) Deduce Uhlmann's theorem from (i), (ii), and part (b). Also conclude that $F(\rho,\sigma) = F(\sigma,\rho)$.

Problem 3: Monotonicity and Contractivity under Quantum Channels

A key property of physically meaningful distinguishability measures is that noise should never make quantum states more distinguishable. In this problem, you will prove that quantum channels decrease trace distance (contractivity) and do not decrease fidelity (monotonicity), and then verify both properties numerically.

a) Contractivity of trace distance.

Let $\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger$ be a trace-preserving quantum operation ($\sum_k M_k^\dagger M_k = I$). Use the POVM-element characterization from Problem 1(b) to prove that

$$D\big(\mathcal{E}(\rho), \mathcal{E}(\sigma)\big) \leq D(\rho, \sigma).$$

Hint[2].

b) Monotonicity of fidelity.

Prove that for any trace-preserving quantum operation $\mathcal{E}$,

$$F\big(\mathcal{E}(\rho), \mathcal{E}(\sigma)\big) \geq F(\rho, \sigma)$$

by using Uhlmann's theorem (Problem 2(c)) together with

$$\mathcal{E}(\tau) = \mathrm{Tr}_E\big(U(\tau \otimes |0\rangle\langle 0|_E)U^\dagger\big)$$

for some unitary U on $Q \otimes E$. Hint[3].

c) Numerical verification.

Consider the complete dephasing channel in the computational basis (equivalently, the phase-flip channel with probability $p = \frac{1}{2}$):

$$\mathcal{E}(\rho) = \tfrac{1}{2}(\rho + Z\rho Z).$$

Apply this channel to the two states from Problem 1(a):

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \qquad \sigma = |+\rangle\langle +| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Compute $D(\mathcal{E}(\rho), \mathcal{E}(\sigma))$ and $F(\mathcal{E}(\rho), \mathcal{E}(\sigma))$, and verify explicitly that

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma), \qquad F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma).$$

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[2] Start from the characterization in Problem 1(b). For a POVM element E on the output space, define

$$\widetilde{E} = \sum_k M_k^\dagger E M_k.$$

Show that $\widetilde{E}$ is again a POVM element, i.e. $0 \leq \widetilde{E} \leq I$. For positivity, test $\widetilde{E}$ on an arbitrary vector $|\psi\rangle$.

[3] Take purifications $|\psi\rangle_{QR}$ of ρ and $|\phi\rangle_{QR}$ of σ. Add an extra system E in the state $|0\rangle_E$, apply $U_{QE} \otimes I_R$, and compare the overlap before and after.

Problem 4: Relating Trace Distance and Fidelity

The trace distance and fidelity are two different ways to quantify how distinguishable two quantum states are. The standard relation between them is given by the *Fuchs–van de Graaf inequalities*:

$$1 - F(\rho, \sigma) \;\leq\; D(\rho, \sigma) \;\leq\; \sqrt{1 - F(\rho, \sigma)^2}.$$

In this problem you will prove the *upper bound*

$$D(\rho, \sigma) \;\leq\; \sqrt{1 - F(\rho, \sigma)^2},$$

and show that it is tight for pure states. The lower bound

$$1 - F(\rho, \sigma) \;\leq\; D(\rho, \sigma)$$

is stated here without proof.

a) Pure states.

For pure states $|a\rangle$ and $|b\rangle$, show that

$$D(|a\rangle, |b\rangle) = \sqrt{1 - F(|a\rangle, |b\rangle)^2}.$$

b) Upper bound: $D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}$.

i) Let X be a Hermitian operator on $Q \otimes R$. Show that

$$\mathrm{Tr}\big|\mathrm{Tr}_R(X)\big| \leq \mathrm{Tr}\,|X|.$$

ii) Use part (i), together with Uhlmann's theorem and part (a), to prove that

$$D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}$$

for arbitrary density operators ρ and σ.