University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 10

This problem set will be discussed on **Apr. 22**.

Problem 1: Syndrome Measurement for the Bit Flip Code

The three-qubit bit flip code protects a single qubit against bit flip (X) errors by encoding

$$|0\rangle \to |0_L\rangle \equiv |000\rangle, \qquad |1\rangle \to |1_L\rangle \equiv |111\rangle,$$

so that a general state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is encoded as $|\psi_L\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$. Each of the three qubits is then sent through an independent bit flip channel, which flips a qubit with probability p.

In this problem you work through the syndrome measurement procedure using the parity-check observables Z_1Z_2 and Z_2Z_3, and determine when the code improves the fidelity of storage.

a) The syndrome table.

Error detection in the bit flip code is performed by measuring the two observables $Z_1Z_2 \equiv Z\otimes Z\otimes I$ and $Z_2Z_3 \equiv I \otimes Z \otimes Z$. Each has eigenvalues ± 1.

i) Show that for any computational basis state $|xyz\rangle$,

$$Z_1Z_2\,|xyz\rangle = (-1)^{x\oplus y}\,|xyz\rangle, \qquad Z_2Z_3\,|xyz\rangle = (-1)^{y\oplus z}\,|xyz\rangle,$$

where $\oplus$ denotes addition modulo 2. Thus Z_1Z_2 "compares" qubits 1 and 2, yielding $+1$ if they agree and -1 if they differ.

ii) For each of the four cases (no error, X_1, X_2, X_3 acting on the encoded state), determine the eigenvalues of Z_1Z_2 and Z_2Z_3. Verify that the four syndrome outcomes are distinct.

b) Non-demolition property.

A key requirement for quantum error correction is that the syndrome measurement must not reveal any information about the encoded quantum state (i.e., about α and β).

i) Consider the error-free encoded state $|\psi_L\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$. The projector onto the $+1$ eigenspace of Z_1Z_2 is $P_+ = (|00\rangle\langle 00| + |11\rangle\langle 11|)\otimes I$. Show that $P_+\,|\psi_L\rangle = |\psi_L\rangle$, so that the outcome $+1$ occurs with certainty regardless of α and β, and the post-measurement state is unchanged.

ii) Now consider the state after a bit flip on qubit 1: $X_1\,|\psi_L\rangle = \alpha\,|100\rangle + \beta\,|011\rangle$. Show that measuring Z_1Z_2 yields -1 with certainty and that the state is unchanged. Explain why the measurement reveals information about the *error* but not about the *encoded state*.

c) Fidelity improvement.

Compare the minimum fidelity of the qubit with and without the bit flip code.

i) Without encoding, a qubit $|\psi\rangle$ sent through the bit flip channel has output $\rho' = (1-p)\,|\psi\rangle\langle\psi| + pX\,|\psi\rangle\langle\psi|X$. Show that the minimum fidelity (over all $|\psi\rangle$) is $F_{\min} = \sqrt{1-p}$.

ii) With the three-qubit code followed by ideal syndrome measurement, correction, and decoding, show that the minimum fidelity is

$$F_{\min} = \sqrt{1 - 3p^2 + 2p^3}.$$

For what range of p does the code improve the minimum fidelity?

Hint[1].

[1] Recall that $F(|\psi\rangle, \rho) = \sqrt{\langle\psi|\rho|\psi\rangle}$. Minimize over $|\psi\rangle$ for part i). For part ii), note that successful error correction (zero or one bit flips) recovers the exact state, while two or three bit flips produce a logical X error on the code space after recovery, and hence after decoding.

Problem 2: The Shor Code

The three-qubit bit flip code corrects X errors but not Z errors. To correct phase errors (Z) as well, we need to also protect in the $|+\rangle/|-\rangle$ basis. The Shor code combines both ideas by *concatenation* into a nine-qubit code that corrects *any* single-qubit error — including errors that are not Pauli operators at all. This is possible because of a remarkable phenomenon called *error discretization.*

a) The phase flip code.

i) The bit flip channel is $\mathcal{E}_{\rm BF}(\rho) = (1-p)\rho + pX\rho X$, and the phase flip channel is $\mathcal{E}_{\rm PF}(\rho) = (1-p)\rho + pZ\rho Z$. Using $HXH = Z$, show that $\mathcal{E}_{\rm PF}(\rho) = H\,\mathcal{E}_{\rm BF}(H\rho H)\,H$ and explain what this implies: how can a bit flip code be turned into a phase flip code?

ii) The phase flip code encodes $|0\rangle \to |+++\rangle$ and $|1\rangle \to |---\rangle$, so a general state becomes $\alpha\,|+++\rangle + \beta\,|---\rangle$. Find two two-qubit observables (analogous to Z_1Z_2 and Z_2Z_3 for the bit flip code) that serve as syndrome measurements, and construct the syndrome table for no error and single-qubit errors Z_1, Z_2, Z_3.

Hint[2].

b) The codewords.

The Shor code is constructed by *concatenation*: first encode using the phase flip code, then encode each qubit using the bit flip code.

i) The phase flip code maps $|0\rangle \to |+++\rangle$ and $|1\rangle \to |---\rangle$; each of these three qubits is then protected against bit flips by encoding $|\pm\rangle$ as

$$|+\rangle \to \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \qquad |-\rangle \to \frac{|000\rangle - |111\rangle}{\sqrt{2}}.$$

Write down the full codewords $|0_L\rangle$ and $|1_L\rangle$ for the Shor code.

ii) How many physical qubits are used? How many additional qubits initialized in $|0\rangle$ are required by the standard encoding circuit?

c) Syndrome measurements in the Shor code.

The Shor code uses eight syndrome observables. For bit flips *within* each block of three qubits:

$$Z_1Z_2,\ Z_2Z_3,\ Z_4Z_5,\ Z_5Z_6,\ Z_7Z_8,\ Z_8Z_9.$$

For phase flips *between* blocks, use the two observables

$$X_{1\cdots6} \equiv X_1X_2X_3X_4X_5X_6, \qquad X_{4\cdots9} \equiv X_4X_5X_6X_7X_8X_9.$$

Work through the following two examples.

i) A bit flip on qubit 5 (i.e., X_5 acts on the encoded state). Which syndrome observables yield -1 and which yield $+1$? Identify the error and describe the recovery operation.

ii) A combined bit-and-phase flip X_1Z_1 acts on qubit 1. Determine all eight syndrome values. Show that the bit flip and phase flip syndromes detect the two components independently, and describe the recovery.

[2] Recall that $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$, so X plays the role for the $|\pm\rangle$ basis that Z plays for the $|0\rangle/|1\rangle$ basis. Also use $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$.

Hint[3].

d) Code limitations.

The Shor code corrects any *single-qubit* error. Here you explore what happens when two qubits in the same block are both flipped.

i) Two bit flip errors occur in block 1: X_1X_2 acts on the encoded state. Determine all eight syndrome values. What does the error-correction procedure conclude, and what correction does it apply?

ii) Apply the correction from part i) to the state after the X_1X_2 error. What is the net operation on block 1? Compute its effect on the block codewords $(|000\rangle + |111\rangle)/\sqrt{2}$ and $(|000\rangle - |111\rangle)/\sqrt{2}$, and explain what has happened to the encoded state.

iii) If each qubit independently suffers a bit flip with probability p, what is the probability that a *given block* of three qubits has two or more flips, so that the inner bit-flip correction in that block fails? What is the probability that this happens in *at least one* of the three blocks? Is the single-block result consistent with your result from Problem 1(c)?

Hint[4].

[3] Recall that operators commute or anti-commute with the syndrome observables depending on whether they share Pauli operators on the same qubits: $XZ = -ZX$ but $XX = ZZ = I$. An anti-commutation flips the eigenvalue of the syndrome observable.

[4] For ii), apply the net operation to $|000\rangle$ and $|111\rangle$ separately. Recall that $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$.

Problem 3: Error Discretization

Physical noise rarely produces clean Pauli errors. A realistic channel might rotate a qubit by an arbitrary angle, or weakly entangle it with its environment. In this problem you show that syndrome measurement automatically converts any such continuous error into a discrete, correctable Pauli error.

a) Rotation error

Suppose qubit j of the encoded state $|\psi_L\rangle = \alpha\,|0_L\rangle + \beta\,|1_L\rangle$ undergoes a rotation error: the noise applies

$$R_x^{(j)}(\theta) = \cos\tfrac{\theta}{2}\, I - i \sin\tfrac{\theta}{2}\, X_j,$$

giving

$$R_x^{(j)}(\theta)\,|\psi_L\rangle = \cos\tfrac{\theta}{2}\ |\psi_L\rangle\ -\ i \sin\tfrac{\theta}{2}\ X_j\,|\psi_L\rangle.$$

i) The corrupted state is a superposition of two terms. What syndrome does each term have?

ii) When the syndrome is measured, what are the two possible outcomes, their probabilities, and the post-measurement state in each case?

iii) Apply the appropriate Pauli correction after each outcome. Show that $|\psi_L\rangle$ is recovered *exactly* in both cases, regardless of θ. What has happened to the continuous parameter θ?

b) Correcting Arbitrary Errors

So far, we have looked at discrete bit and phase flips. However, the Shor code in fact corrects *any* single-qubit error, continuous or non-unitary.

Consider a general noise channel acting on a single physical qubit k of the encoded state $\rho_L = |\psi_L\rangle\langle\psi_L|$:

$$\mathcal{E}(\rho_L) = \sum_j M_j\,\rho_L\,M_j^\dagger,$$

where each Kraus operator acts nontrivially only on qubit k, and $\sum_j M_j^\dagger M_j = I$. Here the physical qubit k is fixed throughout: the noise acts on one specified qubit, so the relevant Pauli error set is $\{I, X_k, Y_k, Z_k\}$ for that fixed k.

Prove that measuring the error syndrome and applying the corresponding Pauli correction perfectly recovers the original state $|\psi_L\rangle$. Hint[5].

[5] Recall that $\{I, X, Y, Z\}$ forms a basis for all 2×2 matrices. Start by expanding each Kraus operator as $M_j = \alpha_j I + \beta_j X_k + \gamma_j Y_k + \delta_j Z_k$. Define the full syndrome projection operator P_E corresponding to a specific Pauli error $E \in \{I, X_k, Y_k, Z_k\}$, and show how it acts on the terms $M_j\,|\psi_L\rangle$.