University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 11

This problem set will be discussed on **Apr. 29**.

Problem 1: Properties of Classical Entropy

The Shannon entropy of a random variable X with probability distribution $p_1, \ldots, p_n$ is defined by

$$H(X) \equiv -\sum_x p_x \log p_x,$$

where logarithms are base two and $0 \log 0 \equiv 0$. A second fundamental quantity is the *relative entropy* of two distributions $p(x)$ and $q(x)$ over the same index set:

$$H(p \,\|\, q) \equiv \sum_x p(x) \log \frac{p(x)}{q(x)},$$

with the standard conventions $0 \log \frac{0}{q} = 0$ for any $q \geq 0$, and $p \log \frac{p}{0} = +\infty$ for $p > 0$. In this problem you develop key inequalities for classical entropy using the relative entropy as a tool, and then apply them to a concrete channel.

a) Non-negativity of the relative entropy.

i) The inequality $\log x \leq (x-1)/\ln 2$ holds for all $x > 0$, with equality if and only if $x = 1$. Use this to prove that $H(p \,\|\, q) \geq 0$, with equality if and only if $p(x) = q(x)$ for all x.

ii) Let X be a random variable with d outcomes. By choosing $q(x) = 1/d$ in part i), show that $H(X) \leq \log d$, with equality if and only if X is uniformly distributed.

b) Subadditivity.

For two random variables X and Y with joint distribution $p(x, y)$, the *joint entropy* is $H(X, Y) = -\sum_{x,y} p(x, y) \log p(x, y)$, and the *mutual information* is

$$I(X : Y) \equiv H(X) + H(Y) - H(X, Y).$$

i) Show that the mutual information is a relative entropy:

$$I(X : Y) = H\big(p(x, y) \,\big\|\, p(x)\, p(y)\big).$$

ii) Deduce the *subadditivity inequality* $H(X, Y) \leq H(X) + H(Y)$, with equality if and only if X and Y are independent.

c) A noisy channel.

Alice sends a bit $X \in \{0, 1\}$ uniformly at random through a *binary symmetric channel* with error probability $\varepsilon = 1/4$: the output Y equals X with probability $3/4$ and is flipped with probability $1/4$.

i) Write down the joint distribution $p(x, y)$ in a table and find the marginal distribution $p(y)$. Compute $H(X)$ and $H(Y)$.

ii) The *conditional entropy* $H(Y|X) \equiv H(X,Y) - H(X)$ measures how uncertain Y remains once X is known. Compute $H(Y|X)$, $H(X,Y)$, and $I(X:Y)$, expressing your answers in terms of the binary entropy $h_{\rm b}(p) = -p\log p - (1-p)\log(1-p)$.

iii) Verify the subadditivity inequality for this example and interpret the value of $I(X:Y)$.

Hint[1].

Problem 2: Von Neumann Entropy

The von Neumann entropy of a quantum state ρ with eigenvalues $\lambda_1, \ldots, \lambda_d$ is

$$S(\rho) \equiv -\operatorname{Tr}(\rho \log \rho) = -\sum_x \lambda_x \log \lambda_x,$$

where again logarithms are base two and $0 \log 0 \equiv 0$. In this problem you explore how the quantum entropy differs from the classical Shannon entropy, prove foundational results, and discover a striking property with no classical analogue.

For a bipartite state ρ^{AB} we use the shorthand

$$S(A,B) \equiv S(\rho^{AB}), \qquad S(A) \equiv S(\rho^{A}), \qquad S(B) \equiv S(\rho^{B}),$$

where $\rho^A = \operatorname{Tr}_B \rho^{AB}$ and $\rho^B = \operatorname{Tr}_A \rho^{AB}$ are the reduced states.

a) Quantum vs. classical entropy.

Alice prepares the ensemble $\{p_i,\, |\psi_i\rangle\}$ with

$$p_0 = \tfrac{1}{2},\ |\psi_0\rangle = |0\rangle, \qquad p_1 = \tfrac{1}{2},\ |\psi_1\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

i) Compute the density matrix $\rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|$.

ii) Find the eigenvalues of ρ and compute $S(\rho)$.

iii) Compare $S(\rho)$ to the Shannon entropy $H(p_0, p_1) = h_{\rm b}(1/2) = 1$ of the preparation probabilities. Is $S(\rho) = H(p_0, p_1)$? Explain why, referring to the overlap $\langle 0|+\rangle$.

b) The joint entropy theorem.

Let p_i be probabilities, $\{|i\rangle\}$ an orthonormal basis for system A, and ρ_i density operators for system B. Prove the *joint entropy theorem*:

$$S\left(\sum_i p_i\, |i\rangle\langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i\, S(\rho_i).$$

Hint[2].

[1] For part ii), first use $p(x,y) = p(x)\, p(y|x)$ to show that

$$H(Y|X) = H(X,Y) - H(X) = -\sum_{x,y} p(x,y) \log p(y|x) = \sum_x p(x)\, H(Y|X=x),$$

where $H(Y|X=x)$ is the Shannon entropy of the conditional distribution of Y given $X = x$. Then note that given $X = x$, the output Y is a binary random variable with $P(Y = x \,|\, X = x) = 3/4$.

[2] Let $\lambda_j^{(i)}$, $\left|e_j^{(i)}\right\rangle$ be the eigenvalues and eigenvectors of ρ_i. Show that $p_i \lambda_j^{(i)}$ and $|i\rangle \left|e_j^{(i)}\right\rangle$ are the eigenvalues and eigenvectors of the composite state.

c) Equal entropies for purifications.

Let $|\psi_{AB}\rangle$ be a pure state of a bipartite system with Schmidt decomposition $|\psi_{AB}\rangle = \sum_i \sqrt{\lambda_i}\ |i_A\rangle\,|i_B\rangle$. Show that $S(A) = S(B)$.

d) Negative conditional entropy.

For a bipartite quantum state ρ^{AB}, the conditional entropy is defined by $S(A|B) \equiv S(A,B) - S(B)$.

i) Consider the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Compute $S(A,B)$, $S(A)$, $S(B)$, and $S(A|B)$.

ii) In classical information theory, $H(X|Y) = H(X,Y) - H(Y) \geq 0$ always. Briefly explain why the quantum conditional entropy can be negative, and what this implies about the correlations between A and B.

Problem 3: Entropic Uncertainty Principle

The Heisenberg uncertainty principle bounds the product $\Delta(C)\,\Delta(D)$ of standard deviations. An alternative formulation uses entropy to capture the idea that one cannot simultaneously know the outcomes of two incompatible measurements.

Let $C = \sum_c c\,|c\rangle\langle c|$ and $D = \sum_d d\,|d\rangle\langle d|$ be two nondegenerate observables, with orthonormal eigenbases $\{|c\rangle\}$ and $\{|d\rangle\}$, respectively. For a state $|\psi\rangle$, let

$$p(c) = |\langle c|\psi\rangle|^2, \qquad q(d) = |\langle d|\psi\rangle|^2$$

be the measurement probability distributions, with corresponding entropies $H(C)$ and $H(D)$. Define the *maximum fidelity*

$$f(C,D) \equiv \max_{c,d} |\langle c|d\rangle|.$$

A sharp entropic uncertainty relation states that

$$H(C) + H(D) \geq 2\log\frac{1}{f(C,D)}.$$

In this problem you prove the weaker but elementary bound

$$H(C) + H(D) \geq -2\log\frac{1+f(C,D)}{2}.$$

a) Fix eigenstates $|c\rangle$ of C and $|d\rangle$ of D, and let $\left|\widetilde{\psi}\right\rangle$ be the projection of $|\psi\rangle$ into the subspace spanned by $|c\rangle$ and $|d\rangle$, with $\lambda = \|\left|\widetilde{\psi}\right\rangle\| \leq 1$. If $\lambda = 0$ or $|\langle c|d\rangle| = 1$, the desired bound is immediate. Assume henceforth that $\lambda > 0$ and $|\langle c|d\rangle| < 1$. Choose the phases of $|c\rangle$ and $|d\rangle$ so that $\langle c|d\rangle$ is real and nonnegative. Choose a unit vector $|d_\perp\rangle$ in this subspace orthogonal to $|d\rangle$, and write

$$|c\rangle = \cos\theta\ |d\rangle + \sin\theta\ |d_\perp\rangle, \qquad \frac{\left|\widetilde{\psi}\right\rangle}{\lambda} = \cos\varphi\ |d\rangle + e^{i\eta}\sin\varphi\ |d_\perp\rangle$$

for some angles $\theta, \varphi \in [0, \pi/2]$ and phase η. Show that

$$p(c)\,q(d) \leq \lambda^4\cos^2(\theta-\varphi)\cos^2\varphi,$$

and conclude that

$$p(c)\,q(d) \leq \left(\frac{1+|\langle c|d\rangle|}{2}\right)^2.$$

Hint[3].

b) Using the bound from part a) and the fact that $-\log$ is decreasing, prove that

$$H(C)+H(D) \geq -2\log\frac{1+f(C,D)}{2}.$$

Hint[4].

c) Let us apply this principle to a concrete example. Consider a single qubit, and let our two incompatible observables be the Pauli matrices $C=Z$ and $D=X$. Assume the qubit is prepared in the state $|\psi\rangle=|0\rangle$.

- **i)** Find the maximum fidelity $f(Z,X)$ between the eigenbases of these two observables.
- **ii)** Calculate the theoretical lower bound for the sum of the entropies $H(Z)+H(X)$ given by the weaker formula you derived in part b).
- **iii)** Calculate the actual probability distributions $p(c)$ and $q(d)$ for the state $|\psi\rangle=|0\rangle$, and compute their respective Shannon entropies $H(Z)$ and $H(X)$.
- **iv)** Verify that the entropic uncertainty bound from part b) holds for this state.

[3] Note that $\langle c|\psi\rangle = \left\langle c\middle|\widetilde{\psi}\right\rangle$ and $\langle d|\psi\rangle = \left\langle d\middle|\widetilde{\psi}\right\rangle$ since $|c\rangle$ and $|d\rangle$ lie in the same two-dimensional subspace. Also,

$$\left|\cos\theta\cos\varphi + e^{i\eta}\sin\theta\sin\varphi\right| \leq \cos\theta\cos\varphi + \sin\theta\sin\varphi = \cos(\theta-\varphi).$$

Here we use that $\theta,\varphi\in[0,\pi/2]$, so all sine and cosine factors are nonnegative.

[4] Note that $H(C) = -\sum_c p(c)\log p(c) = -\sum_{c,d} p(c)\,q(d)\log p(c)$, since $\sum_d q(d)=1$.