University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 11

This problem set will be discussed on **Apr. 29**.

Problem 1: Properties of Classical Entropy

The Shannon entropy of a random variable X with probability distribution $p_1, \ldots, p_n$ is defined by

$$H(X) \equiv -\sum_x p_x \log p_x,$$

where logarithms are base two and $0 \log 0 \equiv 0$. A second fundamental quantity is the *relative entropy* of two distributions $p(x)$ and $q(x)$ over the same index set:

$$H(p \,\|\, q) \equiv \sum_x p(x) \log \frac{p(x)}{q(x)},$$

with the standard conventions $0 \log \frac{0}{q} = 0$ for any $q \geq 0$, and $p \log \frac{p}{0} = +\infty$ for $p > 0$. In this problem you develop key inequalities for classical entropy using the relative entropy as a tool, and then apply them to a concrete channel.

a) Non-negativity of the relative entropy.

i) The inequality $\log x \leq (x-1)/\ln 2$ holds for all $x > 0$, with equality if and only if $x = 1$. Use this to prove that $H(p \,\|\, q) \geq 0$, with equality if and only if $p(x) = q(x)$ for all x.

ii) Let X be a random variable with d outcomes. By choosing $q(x) = 1/d$ in part i), show that $H(X) \leq \log d$, with equality if and only if X is uniformly distributed.

Solution: i) If there exists some x with $p(x) > 0$ and $q(x) = 0$, then by the convention above $H(p \,\|\, q) = +\infty$, so the claim is immediate. Therefore assume that $q(x) > 0$ whenever $p(x) > 0$. Then

$$H(p \,\|\, q) = -\sum_{x:p(x)>0} p(x) \log \frac{q(x)}{p(x)}.$$

Applying $\log x \leq (x-1)/\ln 2$ with $x = q(x)/p(x)$ gives

$$\begin{aligned} -H(p \,\|\, q) &\leq \frac{1}{\ln 2} \sum_{x:p(x)>0} p(x) \left(\frac{q(x)}{p(x)} - 1 \right) \\ &= \frac{1}{\ln 2} \left(\sum_{x:p(x)>0} q(x) - \sum_{x:p(x)>0} p(x) \right) \\ &= \frac{1}{\ln 2} \left(\sum_{x:p(x)>0} q(x) - 1 \right) \\ &\leq 0. \end{aligned}$$

Therefore $H(p \,\|\, q) \geq 0$.

Equality requires equality in both steps above. First, we must have $q(x)/p(x) = 1$ for every x with $p(x) > 0$, that is, $p(x) = q(x)$ on the support of p. Second, we must have $\sum_{x:p(x)>0} q(x) = 1$.

Since q is normalized, this implies

$$\sum_{x:p(x)=0} q(x) = 1 - \sum_{x:p(x)>0} q(x) = 0,$$

so $q(x) = 0$ whenever $p(x) = 0$. Hence $p(x) = q(x)$ for all x.

ii) Setting $q(x) = 1/d$:

$$H(p \,\|\, 1/d) = \sum_x p(x) \log \frac{p(x)}{1/d} = \sum_x p(x) \log p(x) + \log d = -H(X) + \log d.$$

By part i), $-H(X) + \log d \geq 0$, so $H(X) \leq \log d$.

Equality holds iff $p(x) = 1/d$ for all x, i.e. X is uniform.

b) Subadditivity.

For two random variables X and Y with joint distribution $p(x, y)$, the *joint entropy* is $H(X, Y) = -\sum_{x,y} p(x, y) \log p(x, y)$, and the *mutual information* is

$$I(X : Y) \equiv H(X) + H(Y) - H(X, Y).$$

i) Show that the mutual information is a relative entropy:

$$I(X : Y) = H\big(p(x, y) \,\big\|\, p(x)\, p(y)\big).$$

ii) Deduce the *subadditivity inequality* $H(X, Y) \leq H(X) + H(Y)$, with equality if and only if X and Y are independent.

Solution: i)

$$\begin{aligned} H\big(p(x, y) \,\big\|\, p(x)\, p(y)\big) &= \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p(x)\, p(y)} \\ &= \sum_{x,y} p(x, y) \log p(x, y) - \sum_{x,y} p(x, y) \log p(x) - \sum_{x,y} p(x, y) \log p(y). \end{aligned}$$

Using $\sum_y p(x, y) = p(x)$ and $\sum_x p(x, y) = p(y)$, the last two sums become $-\sum_x p(x) \log p(x) = H(X)$ and $-\sum_y p(y) \log p(y) = H(Y)$. Therefore

$$H\big(p(x, y) \,\big\|\, p(x)\, p(y)\big) = -H(X, Y) + H(X) + H(Y) = I(X : Y).$$

ii) By part i) and the non-negativity of relative entropy proven in the previous subproblem, $I(X : Y) \geq 0$, so

$$H(X) + H(Y) - H(X, Y) \geq 0 \quad \Longrightarrow \quad H(X, Y) \leq H(X) + H(Y).$$

Equality holds iff $p(x, y) = p(x)\, p(y)$ for all x, y, i.e. X and Y are independent.

c) A noisy channel.

Alice sends a bit $X \in \{0, 1\}$ uniformly at random through a *binary symmetric channel* with error probability $\varepsilon = 1/4$: the output Y equals X with probability $3/4$ and is flipped with probability $1/4$.

i) Write down the joint distribution $p(x, y)$ in a table and find the marginal distribution $p(y)$. Compute $H(X)$ and $H(Y)$.

ii) The *conditional entropy* $H(Y|X) \equiv H(X,Y) - H(X)$ measures how uncertain Y remains once X is known. Compute $H(Y|X)$, $H(X,Y)$, and $I(X:Y)$, expressing your answers in terms of the binary entropy $h_{\mathrm{b}}(p) = -p\log p - (1-p)\log(1-p)$.

iii) Verify the subadditivity inequality for this example and interpret the value of $I(X:Y)$.

Hint[1].

Solution: **i)** The joint probabilities are $p(x,y) = P(X=x)\,P(Y=y\,|\,X=x)$. Since $P(X=0) = P(X=1) = 1/2$:

$$\begin{aligned} p(0,0) &= P(X=0)\,P(Y=0\,|\,X=0) = \tfrac{1}{2}\cdot\tfrac{3}{4} = \tfrac{3}{8},\\ p(0,1) &= P(X=0)\,P(Y=1\,|\,X=0) = \tfrac{1}{2}\cdot\tfrac{1}{4} = \tfrac{1}{8},\\ p(1,0) &= P(X=1)\,P(Y=0\,|\,X=1) = \tfrac{1}{2}\cdot\tfrac{1}{4} = \tfrac{1}{8},\\ p(1,1) &= P(X=1)\,P(Y=1\,|\,X=1) = \tfrac{1}{2}\cdot\tfrac{3}{4} = \tfrac{3}{8}. \end{aligned}$$

Summarized in a table:

$p(x,y)$	$Y=0$	$Y=1$	$p(x)$
$X=0$	3/8	1/8	1/2
$X=1$	1/8	3/8	1/2
$p(y)$	1/2	1/2	

The marginals are $p(Y=0) = 3/8 + 1/8 = 1/2$ and $p(Y=1) = 1/8 + 3/8 = 1/2$, so Y is also uniform. Both X and Y are uniform over two outcomes, so

$$H(X) = -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{2} = 1 \text{ bit}, \qquad H(Y) = 1 \text{ bit (same calculation)}.$$

ii) Starting from the definition of conditional entropy,

$$\begin{aligned} H(Y|X) &= H(X,Y) - H(X)\\ &= -\sum_{x,y} p(x,y)\log p(x,y) + \sum_x p(x)\log p(x)\\ &= -\sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)}\\ &= -\sum_{x,y} p(x,y)\log p(y|x)\\ &= \sum_x p(x)\Big(-\sum_y p(y|x)\log p(y|x)\Big)\\ &= \sum_x p(x)\,H(Y|X=x). \end{aligned}$$

[1] For part ii), first use $p(x,y) = p(x)\,p(y|x)$ to show that

$$H(Y|X) = H(X,Y) - H(X) = -\sum_{x,y} p(x,y)\log p(y|x) = \sum_x p(x)\,H(Y|X=x),$$

where $H(Y|X=x)$ is the Shannon entropy of the conditional distribution of Y given $X=x$. Then note that given $X=x$, the output Y is a binary random variable with $P(Y=x\,|\,X=x) = 3/4$.

Given $X = x$, the conditional distribution of Y is the same for both values of x: $P(Y = x \,|\, X = x) = 3/4$ and $P(Y \neq x \,|\, X = x) = 1/4$. So for either value of x, the variable Y has a two-outcome distribution with probabilities $3/4$ and $1/4$, and hence

$$H(Y|X = x) = h_{\rm b}(1/4) \quad (= h_{\rm b}(3/4)).$$

Therefore

$$H(Y|X) = \tfrac{1}{2}\, h_{\rm b}(1/4) + \tfrac{1}{2}\, h_{\rm b}(1/4) = h_{\rm b}(1/4).$$

Using the defining relation $H(X,Y) = H(X) + H(Y|X)$,

$$H(X,Y) = 1 + h_{\rm b}(1/4).$$

The mutual information is

$$I(X:Y) = H(X) + H(Y) - H(X,Y) = 1 + 1 - (1 + h_{\rm b}(1/4)) = 1 - h_{\rm b}(1/4).$$

Numerically, $h_{\rm b}(1/4) \approx 0.811$ bits, so $I(X:Y) \approx 0.189$ bits.

iii) Subadditivity check:

$$H(X,Y) = 1 + h_{\rm b}(1/4) \approx 1.811 \;\leq\; 2 = H(X) + H(Y). \quad \checkmark$$

The gap is $I(X:Y) \approx 0.189 > 0$, reflecting the correlation between input and output: seeing Y gives partial (but not complete) information about X. The mutual information quantifies how many bits per channel use Bob learns about Alice's message.

Problem 2: Von Neumann Entropy

The von Neumann entropy of a quantum state ρ with eigenvalues $\lambda_1, \ldots, \lambda_d$ is

$$S(\rho) \equiv -\operatorname{Tr}(\rho \log \rho) = -\sum_x \lambda_x \log \lambda_x,$$

where again logarithms are base two and $0 \log 0 \equiv 0$. In this problem you explore how the quantum entropy differs from the classical Shannon entropy, prove foundational results, and discover a striking property with no classical analogue.

For a bipartite state ρ^{AB} we use the shorthand

$$S(A,B) \equiv S(\rho^{AB}), \qquad S(A) \equiv S(\rho^{A}), \qquad S(B) \equiv S(\rho^{B}),$$

where $\rho^A = \operatorname{Tr}_B \rho^{AB}$ and $\rho^B = \operatorname{Tr}_A \rho^{AB}$ are the reduced states.

a) Quantum vs. classical entropy.

Alice prepares the ensemble $\{p_i,\, |\psi_i\rangle\}$ with

$$p_0 = \tfrac{1}{2},\ |\psi_0\rangle = |0\rangle, \qquad p_1 = \tfrac{1}{2},\ |\psi_1\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

- **i)** Compute the density matrix $\rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|$.
- **ii)** Find the eigenvalues of ρ and compute $S(\rho)$.
- **iii)** Compare $S(\rho)$ to the Shannon entropy $H(p_0, p_1) = h_{\rm b}(1/2) = 1$ of the preparation probabilities. Is $S(\rho) = H(p_0, p_1)$? Explain why, referring to the overlap $\langle 0|+\rangle$.

Solution: i)

$$\rho = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|+\rangle\langle +| = \tfrac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \tfrac{1}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}3/4 & 1/4\\ 1/4 & 1/4\end{pmatrix}.$$

ii) The eigenvalues satisfy $\lambda^2 - \lambda + \det\rho = 0$ with $\det\rho = 3/16 - 1/16 = 1/8$. Using the quadratic formula:

$$\lambda_\pm = \frac{1 \pm \sqrt{1 - 1/2}}{2} = \frac{1 \pm 1/\sqrt{2}}{2}.$$

Numerically, $\lambda_+ \approx 0.854$ and $\lambda_- \approx 0.146$. The von Neumann entropy is

$$S(\rho) = h_\mathrm{b}\left(\frac{1 + 1/\sqrt{2}}{2}\right) \approx 0.601 \text{ bits}.$$

iii) $S(\rho) \approx 0.601 < 1 = H(p_0, p_1)$. They are *not* equal.

For an ensemble of pure states, equality with the Shannon entropy occurs when the preparation states are mutually orthogonal. Here $\langle 0|+\rangle = 1/\sqrt{2} \neq 0$, so the states are non-orthogonal. The overlap means that the two preparations are partially indistinguishable, reducing the effective information content of the ensemble below $H(p_0, p_1)$.

b) The joint entropy theorem.

Let p_i be probabilities, $\{|i\rangle\}$ an orthonormal basis for system A, and ρ_i density operators for system B. Prove the *joint entropy theorem*:

$$S\left(\sum_i p_i\,|i\rangle\langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i\,S(\rho_i).$$

Hint[2].

Solution: Let $\rho_i = \sum_j \lambda_j^{(i)} \left|e_j^{(i)}\right\rangle\!\left\langle e_j^{(i)}\right|$ be a spectral decomposition of each ρ_i. Then the composite state is

$$\sigma_{AB} = \sum_i p_i\,|i\rangle\langle i| \otimes \rho_i = \sum_{i,j} p_i\lambda_j^{(i)}\left(|i\rangle\left|e_j^{(i)}\right\rangle\right)\left(\langle i|\left\langle e_j^{(i)}\right|\right).$$

[2] Let $\lambda_j^{(i)}$, $\left|e_j^{(i)}\right\rangle$ be the eigenvalues and eigenvectors of ρ_i. Show that $p_i\lambda_j^{(i)}$ and $|i\rangle\left|e_j^{(i)}\right\rangle$ are the eigenvalues and eigenvectors of the composite state.

Because $\{|i\rangle\}$ is orthonormal on A and $\{\left|e_j^{(i)}\right\rangle\}$ is orthonormal on B for each i, the set $\{|i\rangle \left|e_j^{(i)}\right\rangle\}$ is orthonormal on $A \otimes B$. Therefore $p_i \lambda_j^{(i)}$ are the eigenvalues of σ_{AB}, and

$$\begin{aligned} S(\sigma_{AB}) &= -\sum_{i,j} p_i \lambda_j^{(i)} \log\bigl(p_i \lambda_j^{(i)}\bigr) \\ &= -\sum_{i,j} p_i \lambda_j^{(i)} \log p_i - \sum_{i,j} p_i \lambda_j^{(i)} \log \lambda_j^{(i)} \\ &= -\sum_i p_i \log p_i \underbrace{\sum_j \lambda_j^{(i)}}_{=\mathrm{Tr}(\rho_i)=1} + \sum_i p_i \underbrace{\left(-\sum_j \lambda_j^{(i)} \log \lambda_j^{(i)}\right)}_{=S(\rho_i)} \\ &= H(p_i) + \sum_i p_i\, S(\rho_i). \qquad \checkmark \end{aligned}$$

c) Equal entropies for purifications.

Let $|\psi_{AB}\rangle$ be a pure state of a bipartite system with Schmidt decomposition $|\psi_{AB}\rangle = \sum_i \sqrt{\lambda_i}\ |i_A\rangle\, |i_B\rangle$. Show that $S(A) = S(B)$.

Solution: From the Schmidt decomposition, the reduced density operators are

$$\rho^A = \mathrm{Tr}_B\, |\psi_{AB}\rangle\langle\psi_{AB}| = \sum_i \lambda_i\, |i_A\rangle\langle i_A|, \qquad \rho^B = \mathrm{Tr}_A\, |\psi_{AB}\rangle\langle\psi_{AB}| = \sum_i \lambda_i\, |i_B\rangle\langle i_B|.$$

Both have exactly the same set of eigenvalues $\{\lambda_i\}$. Since the von Neumann entropy depends only on the eigenvalues, $S(A) = S(B)$.

d) Negative conditional entropy.

For a bipartite quantum state ρ^{AB}, the conditional entropy is defined by $S(A|B) \equiv S(A,B) - S(B)$.

- **i)** Consider the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Compute $S(A,B)$, $S(A)$, $S(B)$, and $S(A|B)$.
- **ii)** In classical information theory, $H(X|Y) = H(X,Y) - H(Y) \geq 0$ always. Briefly explain why the quantum conditional entropy can be negative, and what this implies about the correlations between A and B.

Solution: i) $|\Phi^+\rangle$ is a pure state of AB. A pure state $\rho = |\psi\rangle\langle\psi|$ has eigenvalues 1 and 0 (with the eigenvector for eigenvalue 1 being $|\psi\rangle$ itself), so $S = -1 \cdot \log 1 - 0 \cdot \log 0 = 0$. Therefore

$$S(A,B) = 0.$$

The reduced state of each qubit is the maximally mixed state:

$$\rho^A = \rho^B = \frac{I}{2},$$

so $S(A) = S(B) = \log 2 = 1$ bit. (This also follows from the previous subproblem: $S(A) = S(B)$ for any pure bipartite state.)

The conditional entropy is

$$S(A|B) = S(A,B) - S(B) = 0 - 1 = -1.$$

ii) Classically, conditioning can only reduce or maintain uncertainty: knowing Y never makes X *more* uncertain. This fails in quantum mechanics because of *entanglement*. For the Bell state, the joint system is in a pure state (zero uncertainty), yet each subsystem individually is maximally mixed (maximum uncertainty). For example, if both qubits are measured in the computational basis, the outcomes are perfectly correlated, but neither subsystem carries any local information.

In fact, for a pure bipartite state, $S(A|B) < 0$ if and only if the state is entangled: since $S(A,B) = 0$ for a pure state, we have $S(A|B) = -S(B)$, which is negative exactly when the reduced state is mixed.

Problem 3: Entropic Uncertainty Principle

The Heisenberg uncertainty principle bounds the product $\Delta(C)\,\Delta(D)$ of standard deviations. An alternative formulation uses entropy to capture the idea that one cannot simultaneously know the outcomes of two incompatible measurements.

Let $C = \sum_c c\,|c\rangle\langle c|$ and $D = \sum_d d\,|d\rangle\langle d|$ be two nondegenerate observables, with orthonormal eigenbases $\{|c\rangle\}$ and $\{|d\rangle\}$, respectively. For a state $|\psi\rangle$, let

$$p(c) = |\langle c|\psi\rangle|^2, \qquad q(d) = |\langle d|\psi\rangle|^2$$

be the measurement probability distributions, with corresponding entropies $H(C)$ and $H(D)$. Define the *maximum fidelity*

$$f(C,D) \equiv \max_{c,d} |\langle c|d\rangle|.$$

A sharp entropic uncertainty relation states that

$$H(C) + H(D) \geq 2\log\frac{1}{f(C,D)}.$$

In this problem you prove the weaker but elementary bound

$$H(C) + H(D) \geq -2\log\frac{1+f(C,D)}{2}.$$

a) Fix eigenstates $|c\rangle$ of C and $|d\rangle$ of D, and let $\left|\widetilde{\psi}\right\rangle$ be the projection of $|\psi\rangle$ into the subspace spanned by $|c\rangle$ and $|d\rangle$, with $\lambda = \|\left|\widetilde{\psi}\right\rangle\| \leq 1$. If $\lambda = 0$ or $|\langle c|d\rangle| = 1$, the desired bound is immediate. Assume henceforth that $\lambda > 0$ and $|\langle c|d\rangle| < 1$. Choose the phases of $|c\rangle$ and $|d\rangle$ so that $\langle c|d\rangle$ is real and nonnegative. Choose a unit vector $|d_\perp\rangle$ in this subspace orthogonal to $|d\rangle$, and write

$$|c\rangle = \cos\theta\;|d\rangle + \sin\theta\;|d_\perp\rangle, \qquad \frac{\left|\widetilde{\psi}\right\rangle}{\lambda} = \cos\varphi\;|d\rangle + e^{i\eta}\sin\varphi\;|d_\perp\rangle$$

for some angles $\theta, \varphi \in [0, \pi/2]$ and phase η. Show that

$$p(c)\,q(d) \leq \lambda^4\cos^2(\theta-\varphi)\cos^2\varphi,$$

and conclude that

$$p(c)\,q(d) \leq \left(\frac{1+|\langle c|d\rangle|}{2}\right)^2.$$

Hint[3].

Solution: If $\lambda = 0$, then $\left|\widetilde{\psi}\right\rangle = 0$, so $\langle c|\psi\rangle = \langle d|\psi\rangle = 0$ and hence $p(c)q(d) = 0$. The desired inequality is immediate. If $|\langle c|d\rangle| = 1$, then $|c\rangle$ and $|d\rangle$ differ only by a phase, so the right-hand side is 1 and therefore

$$p(c)q(d) \le 1 = \left(\frac{1+|\langle c|d\rangle|}{2}\right)^2.$$

Thus it remains to consider the case $\lambda > 0$ and $|\langle c|d\rangle| < 1$.

Since $|c\rangle$ and $|d\rangle$ both lie in the two-dimensional subspace, projecting $|\psi\rangle$ onto this subspace does not change the relevant inner products: $\langle c|\psi\rangle = \left\langle c\middle|\widetilde{\psi}\right\rangle$ and $\langle d|\psi\rangle = \left\langle d\middle|\widetilde{\psi}\right\rangle$. After choosing phases of $|c\rangle$ and $|d\rangle$ so that $\langle c|d\rangle$ is real and nonnegative, we may take the parametrization with $\theta, \varphi \in [0, \pi/2]$. Then

$$|\langle c|d\rangle| = \langle c|d\rangle = \cos\theta.$$

Write

$$|c\rangle = \cos\theta\ |d\rangle + \sin\theta\ |d_\perp\rangle, \qquad \frac{\left|\widetilde{\psi}\right\rangle}{\lambda} = \cos\varphi\ |d\rangle + e^{i\eta}\sin\varphi\ |d_\perp\rangle,$$

where $|d_\perp\rangle$ is orthogonal to $|d\rangle$ in the same subspace. Then

$$q(d) = |\langle d|\psi\rangle|^2 = |\left\langle d\middle|\widetilde{\psi}\right\rangle|^2 = \lambda^2\cos^2\varphi.$$

Also,

$$\left\langle c\middle|\widetilde{\psi}\right\rangle = \lambda\big(\cos\theta\cos\varphi + e^{i\eta}\sin\theta\sin\varphi\big),$$

so

$$p(c) = |\langle c|\psi\rangle|^2 = \lambda^2\big|\cos\theta\cos\varphi + e^{i\eta}\sin\theta\sin\varphi\big|^2 \le \lambda^2\cos^2(\theta-\varphi),$$

where the last step uses $|a + e^{i\eta}b| \le |a| + |b|$ and, since $\theta, \varphi \in [0, \pi/2]$, $\cos\theta\cos\varphi + \sin\theta\sin\varphi = \cos(\theta-\varphi)$. Therefore

$$p(c)\,q(d) \le \lambda^4\cos^2(\theta-\varphi)\cos^2\varphi.$$

Since $\lambda \le 1$, it remains to maximize $\cos^2(\theta-\varphi)\cos^2\varphi$. Using

$$\cos(\theta-\varphi)\cos\varphi = \frac{\cos(\theta-2\varphi)+\cos\theta}{2} \le \frac{1+\cos\theta}{2},$$

we obtain

$$\cos^2(\theta-\varphi)\cos^2\varphi \le \left(\frac{1+\cos\theta}{2}\right)^2.$$

Hence

$$p(c)\,q(d) \le \left(\frac{1+\cos\theta}{2}\right)^2 = \left(\frac{1+|\langle c|d\rangle|}{2}\right)^2.$$

Equality is achieved when $\lambda = 1$, $\eta = 0$, and $\varphi = \theta/2$.

[3] Note that $\langle c|\psi\rangle = \left\langle c\middle|\widetilde{\psi}\right\rangle$ and $\langle d|\psi\rangle = \left\langle d\middle|\widetilde{\psi}\right\rangle$ since $|c\rangle$ and $|d\rangle$ lie in the same two-dimensional subspace. Also,

$$\big|\cos\theta\cos\varphi + e^{i\eta}\sin\theta\sin\varphi\big| \le \cos\theta\cos\varphi + \sin\theta\sin\varphi = \cos(\theta-\varphi).$$

Here we use that $\theta, \varphi \in [0, \pi/2]$, so all sine and cosine factors are nonnegative.

b) Using the bound from part a) and the fact that $-\log$ is decreasing, prove that

$$H(C) + H(D) \geq -2 \log \frac{1 + f(C,D)}{2}.$$

Hint[4].

Solution: Using the hint and that $\log p(c) + \log q(d) = \log\big(p(c)\, q(d)\big)$, we get

$$H(C) + H(D) = -\sum_{c,d} p(c)\, q(d) \log\big(p(c)\, q(d)\big).$$

From part a), $p(c)\, q(d) \leq \big(\frac{1+f(C,D)}{2}\big)^2$ for all c, d. Since $-\log$ is decreasing,

$$-\log\big(p(c)\, q(d)\big) \geq -\log\left(\frac{1 + f(C,D)}{2}\right)^2 = -2 \log \frac{1 + f(C,D)}{2}.$$

Multiplying by $p(c)\, q(d) \geq 0$ and summing:

$$H(C) + H(D) \geq -2 \log \frac{1 + f(C,D)}{2} \sum_{c,d} p(c)\, q(d) = -2 \log \frac{1 + f(C,D)}{2}. \qquad \checkmark$$

c) Let us apply this principle to a concrete example. Consider a single qubit, and let our two incompatible observables be the Pauli matrices $C = Z$ and $D = X$. Assume the qubit is prepared in the state $|\psi\rangle = |0\rangle$.

i) Find the maximum fidelity $f(Z, X)$ between the eigenbases of these two observables.

ii) Calculate the theoretical lower bound for the sum of the entropies $H(Z) + H(X)$ given by the weaker formula you derived in part b).

iii) Calculate the actual probability distributions $p(c)$ and $q(d)$ for the state $|\psi\rangle = |0\rangle$, and compute their respective Shannon entropies $H(Z)$ and $H(X)$.

iv) Verify that the entropic uncertainty bound from part b) holds for this state.

Solution: i) The eigenstates of Z are $\{|0\rangle, |1\rangle\}$ and the eigenstates of X are $\{|+\rangle, |-\rangle\}$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The inner product between any Z-eigenstate and any X-eigenstate has a magnitude of exactly $1/\sqrt{2}$. Therefore, the maximum fidelity is

$$f(Z, X) = \frac{1}{\sqrt{2}}.$$

ii) Plugging this fidelity into the bound from part b):

$$\begin{aligned} -2 \log \left(\frac{1 + f(Z,X)}{2}\right) &= -2 \log \left(\frac{1 + 1/\sqrt{2}}{2}\right) \\ &= -2 \log \left(\frac{2 + \sqrt{2}}{4}\right) \approx 0.457. \end{aligned}$$

iii) For the state $|\psi\rangle = |0\rangle$:

[4] Note that $H(C) = -\sum_c p(c) \log p(c) = -\sum_{c,d} p(c)\, q(d) \log p(c)$, since $\sum_d q(d) = 1$.

- **Measuring Z:** The state is already an eigenstate.

$$p(0) = |\langle 0|0\rangle|^2 = 1, \qquad p(1) = |\langle 1|0\rangle|^2 = 0.$$

  The entropy is $H(Z) = -1\log(1) - 0 = 0$ (perfect certainty).

- **Measuring X:** The state has equal overlap with both eigenstates.

$$q(+) = |\langle +|0\rangle|^2 = \frac{1}{2}, \qquad q(-) = |\langle -|0\rangle|^2 = \frac{1}{2}.$$

  The entropy is $H(X) = -\frac{1}{2}\log(1/2) - \frac{1}{2}\log(1/2) = 1$ (maximum uncertainty).

iv) The sum of the actual entropies is $H(Z) + H(X) = 0 + 1 = 1$. Comparing this to our theoretical lower bound:

$$1 \geq 0.457.$$

The inequality holds, verifying the entropic uncertainty bound. The weaker bound is not tight in this example, but it is nevertheless correctly satisfied.