FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2009

Problem set 5

5.1 Harmonic oscillator states

The Hamiltonian of a one-dimensional harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + m^2\omega^2\hat{x}^2) \tag{1}$$

a) Introduce the lowering and raising operators

$$\hat{a} = \sqrt{\frac{1}{2m\omega\hbar}}(m\omega\hat{x} + i\hat{p})\,, \quad \hat{a}^\dagger = \sqrt{\frac{1}{2m\omega\hbar}}(m\omega\hat{x} - i\hat{p}) \tag{2}$$

and show that the Hamiltonian can be written as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})) \tag{3}$$

b) The energy eigenvectors $|n\rangle$ are defined by the equation

$$\hat{H}|n\rangle = E_n|n\rangle \tag{4}$$

Show that $E_n = \hbar\omega(n + \frac{1}{2})$, and that the raising and lowering operators satisfy the relations

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \tag{5}$$

b) In the energy representation (or n representation), the energy eigenvectors $|n\rangle$ are used as a complete, orthonormal basis. A general observable $\hat{A}$ in this basis can be expressed as an infinite matrix with matrix elements

$$A_{mn} = \langle m|\hat{A}|n\rangle \tag{6}$$

Find the expressions for the (m, n) matrix elements for the following operators $\hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2$ and $\hat{x}\hat{p}+\hat{p}\hat{x}$. Write the operators in matrix form with the 4x4 submatrix corresponding to $n = 0, 1, 2, 3$ written out explicitely.

c) All energy eigenstates can be generated from the ground state by use of the relations,

$$\hat{a}|0\rangle = 0\,, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{7}$$

Write these equations in the coordinate representation (x-representation), where the energy eigenstates are represented by wave functions $\psi_n(x) = \langle x|n\rangle$. Use the equations to show that the eigenstates in this representation have the form

$$\psi_n(x) = P_n(x)e^{-\lambda x^2} \tag{8}$$

with $P_n(x)$ as a polynomial of order n in x. Find λ and $P_n(x)$ for the three lowest states, $n = 0, 1, 2$.

5.2 Displacement operators in phase space

For a particle moving in one dimension the position coordinate x and the momentum p define the coordinates of the two-dimensional classical *phase space*.

In the quantum description of the one-dimensional harmonic oscillator non-hermitian lowering operator is defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\,\hat{x} + i\hat{p}) \tag{9}$$

We may consider this as the operator of a complex phase space variable, with position as the real part and momentum as the imaginary part. It has a dimensionless form due to the constants introduced in the expression.

A coherent state, in a similar way is characterized by a complex number z, the eigenvalue of $\hat{a}$, which we may interpret as a complex phase space coordinate,

$$z = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\,x_c + ip_c) \tag{10}$$

The following operator

$$\hat{\mathcal{D}}(z) = e^{(z\hat{a}^\dagger - z^*\hat{a})} \tag{11}$$

acts as a displacement operator in phase space, in the sense

$$\hat{\mathcal{D}}(z)\hat{x}\hat{\mathcal{D}}(z)^\dagger = \hat{x} - x_c\ ,\ \ \hat{\mathcal{D}}(z)\hat{p}\hat{\mathcal{D}}(z)^\dagger = \hat{p} - p_c \tag{12}$$

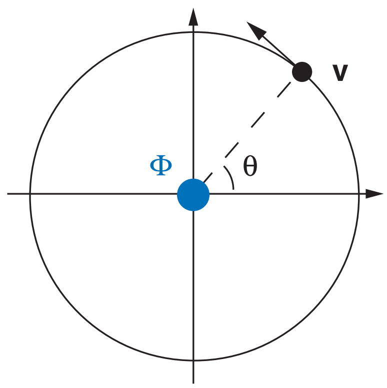
Show that displacements in two different directions in general will not commute but rather satisfy a relation of the form

$$\hat{\mathcal{D}}(z_a)\hat{\mathcal{D}}(z_b) = e^{i\alpha(z_a,z_b)}\hat{\mathcal{D}}(z_b)\hat{\mathcal{D}}(z_a) \tag{13}$$

with $\alpha(z_a, z_b)$ as a complex phase. Determine the phase as a function of z_a and z_b. What is the condition for the two operators to commute?

5.3 Particle encircling a magnetic flux
(Midterm Exam 2004)

A particle with mass m and charge e moves freely on a circle of radius R. Through the circle passes a solenoid that carries a magnet flux Φ. We may consider the total flux to be confined to the solenoid so that the magnetic field vanishes on the circle where the particle moves.



In the following make use of the general expressions for the Hamiltonian of a particle in a magnetic field

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 \tag{14}$$

and for the probability current

$$\mathbf{j} = -\frac{i\hbar}{2m}(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*) - \frac{e}{m}\mathbf{A}\psi^*\psi \tag{15}$$

with $\mathbf{A}$ as the vector potential, so that the magnetic field is $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.

Use in the following the angle variable θ as coordinate for the particle on the circle.

a) Assume rotational invariance about the center of the circle and show that the vector potential on the circle takes the constant value $A = \Phi/2\pi R$ with direction along the circle. Explain why this vector potential has no influence on the motion of the particle when this is described by the classical equations of motion.

b) Express the Hamiltonian as an operator acting on the wave functions $\psi(\theta)$ for the particle on the circle. Find the energy eigenvalues and show that the energy spectrum varies periodically with the flux Φ. What is the flux period Φ_0? Plot the four lowest energies as functions of Φ in the interval from 0 to Φ_0. Characterize the ground state by its angular momentum in the same interval. What is special for the spectrum at $\Phi = \Phi_0/2$?

c) Find the probability current for a general wave function $\psi(\theta)$, and determine the value of the ground state current as a function of Φ. What is the maximum value of the ground state current and what value for the particle velocity does that correspond to.

d) Find the propagator $\mathcal{G}(\theta, t; 0, 0) = \langle \theta, t | 0, 0 \rangle$ expressed as a series expansion in angular momentum states. Show further how this series can be written as a Jacobi theta function for general Φ. Use the definition of the Jacobi theta function as given in Problem 4.3 (Problem Set 4, 2009).

e) For the Lagrangian of a particle in a magnetic field the effect of the vector potential is to add a term proportional to the velocity

$$L = \frac{1}{2} m v^2 + e\mathbf{A} \cdot \mathbf{v} \tag{16}$$

Follow the path integral approach of Problem 3.2 to find the propagator by summing over all classical paths with the given initial and final conditions. In the same way as discussed there the propagator derived in this way is equivalent to the one derived in d). Show this. (Use the properties listed in Problem 3.2 for the Jacobi theta function.)