UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 4110 Non-relativistic quantum mechanics
Day of exam: Thursday, December 4, 2008
Exam hours: 3 hours, beginning at 14:30
This examination paper consists of 3 pages
Permitted materials: Calculator
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before you begin.

PROBLEM 1
Spin half particle in a harmonic oscillator potential
A spin half particle is moving in a one-dimensional harmonic oscillator potential (in the x-direction) under the influence of a constant magnetic field (in the z-direction). The Hamiltonian is

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{1}{2}\hbar\omega_1\sigma_z + \lambda\hbar(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{1}$$

where the first term is the harmonic oscillator part with ω_0 as the oscillator frequency, the second term is the spin energy due to the magnetic field, with ω_1 as the spin precession frequency, and the third term is a coupling term between the spin and the position coordinate of the particle. The spin flip operators are defined as $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hat{a}$ and $\hat{a}^\dagger$ are the standard lowering and raising operators of the harmonic oscillator and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.

When $\lambda = 0$, the spin and position of the particle are uncoupled and the energy eigenstates are $|n, m\rangle$ with $n = 0, 1, 2, ...$ as the harmonic oscillator quantum number and $m = \pm 1$ as the spin quantum number, corresponding to spin up/down along the z-axis. When $\lambda \neq 0$, the unperturbed eigenstates will pairwise be coupled by the Hamiltonian, so that $|n, +1\rangle$ is coupled to $|n + 1, -1\rangle$.

a) Consider the two-dimensional subspace spanned by basis vectors$|0, +1\rangle$ and $|1, -1\rangle$. Show that in this space the Hamiltonian takes the form of a 2x2 matrix which can be written as

$$H = \hbar\Delta \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \hbar\epsilon\mathbb{1} \tag{2}$$

with $\mathbb{1}$ as the 2x2 identity matrix. Determine $\Delta\cos\theta$, $\Delta\sin\theta$ and ϵ.

b) Find the energies and eigenstates of H in the two-dimensional subspace, expressed as functions of Δ, θ and ϵ.

c) The basis vectors $|n, m\rangle$ can be regarded as tensor products of position and spin vectors, $|n, m\rangle = |n\rangle \otimes |m\rangle$. The two eigenstates found under b) will be entangled with respect to the position and spin variables. Determine the degree of entanglement as function of θ. What value for θ gives the smallest what gives the largest degree of entanglement?

PROBLEM 2

Electric dipole transition

We consider the transition in hydrogen from the excited 2p level to the ground state 1s, where a single photon is emitted. The initial atomic state (A) we assume to have $m = 0$ for the z-component of the orbital angular momentum, so that the quantum numbers of this state are $(n, l, m) = (2, 1, 0)$, with n as the principle quantum number and l as the orbital angular momentum quantum number. Similarly the ground state (B) has quantum numbers $(n, l, m) = (1, 0, 0)$. When expressed in polar coordinates the wave functions of the two states (with intrinsic spin of the electron not included) are given by

$$\psi_A(r, \phi, \theta) = \frac{1}{\sqrt{32\pi a_0^3}} \cos\theta \, \frac{r}{a_0} \, e^{-\frac{r}{2a_0}}$$
$$\psi_B(r, \phi, \theta) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}} \tag{3}$$

where a_0 is the Bohr radius.

We remind you about the form of the interaction matrix element in the dipole approximation,

$$\langle B, 1_{\mathbf{k}a} | \hat{H}_{emis} | A, 0 \rangle = ie \sqrt{\frac{\hbar\omega}{2V\epsilon_0}} \, \boldsymbol{\epsilon}^*_{\mathbf{k}a} \cdot \mathbf{r}_{BA} \tag{4}$$

where e is the electron charge, $\mathbf{k}$ is the wave vector of the photon, a is the polarization quantum number, ω is the photon frequency and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is a polarization vector. V is a normalization volume for the electromagnetic wave functions, ϵ_0 is the permittivity of vacuum and $\mathbf{r}_{BA}$ is the matrix element of the electron position operator between the initial and final atomic states.

a) Explain why the x- and y-components of $\mathbf{r}_{BA}$ vanish while the z-component has the form $z_{BA} = \nu a_0$, with ν as a numerical factor. Determine the value of ν. (A useful integration formula is $\int_0^\infty dx \, x^n \, e^{-x} = n!$.)

b) To first order in perturbation theory the interaction matrix element (4) determines the direction of the emitted photon, in the form of a probability distribution $p(\phi, \theta)$, where (ϕ, θ) are the polar angles of the wave vector $\mathbf{k}$. Determine $p(\phi, \theta)$ from the above expressions.

c) The life time of the 2p state is $\tau_{2p} = 1.6 \cdot 10^{-9} s$ while the excited 2s state (with angular momentum $l = 0$) has a much longer life time, $\tau_{2s} = 0.12 s$. Du you have a (qualitative) explanation for the large difference?

PROBLEM 3

Density operators and entanglement

Give a brief and concise discussion of the following points:

a) List the general properties of density operators (or density matrices) and specify the difference between a *pure* and a *mixed* state.

b) For a composite system consisting of two parts $\mathcal{A}$ and $\mathcal{B}$ use the density operator formulation to explain the difference between, uncorrelated states, states with classical correlations (separable states) and entangled states.

c) Assume the full system is in a pure state, described by the state vector $|\psi\rangle$. What is meant by the *Schmidt decomposition* of this state vector relative to the two subsystems $\mathcal{A}$ and $\mathcal{B}$? Use the decomposition to find expressions for the reduced density operators of the two subsystems, and show that the von Neumann entropy of the reduced density operators are equal.