FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2005

The problem set is available from Friday October 14. The set consists of 2 problems written on 4 pages.

Deadline for returning solutions
is Friday October 21.

Return of solutions
The solutions can be returned either in written/printed form or as an e-mail attachment.
Written/printed solutions can be returned at Ekspedisjonskontoret in the Physics Building. Please add a copy that the lecturer can keep for evaluation at the final exam.
E-mailed solutions: Please send the solutions as one file, preferably in pdf format. E-mail address:j.m.leinaas@fys.uio.no.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471).
Language
Solutions may be written in Norwegian or English, depending on your preference.

PROBLEMS

1 Spin motion in an oscillating field.
We study in this problem first spin motion in a constant magnetic field, then the effect of including an additional, oscillating field. Results that are derived in Sect. 1.3.2 of the lecture notes may be used in the solution.

An electron with spin vector

$$\hat{\mathbf{S}} = (\hbar/2)\boldsymbol{\sigma} \tag{1}$$

and magnetic moment

$$\hat{\boldsymbol{\mu}} = \frac{e}{m}\hat{\mathbf{S}} \tag{2}$$

is situated in a constant magnetic field $\mathbf{B} = B_0\mathbf{k}$. The spin motion is assumed to be independent of the orbital motion of the electron.

a) The spin state is described by a time dependent density matrix

$$\rho(t) = \frac{1}{2}(\mathbf{1} + \mathbf{r}(t) \cdot \boldsymbol{\sigma}) \tag{3}$$

As initial condition for the motion we have $\mathbf{r} = \mathbf{r}_0$ for $t = 0$. Give a general expression for $\rho(t)$ in terms of the time evolution operator, and use this to determine the time dependent vector $\mathbf{r(t)}$.

b) Assume next the initial condition $\mathbf{r_0} = a\mathbf{k}$ ($\mathbf{k}$ is the unit vector in the z-direction). What are the allowed values of a? An oscillating field is turned on so that the total magnetic field is

$$\mathbf{B} = B_0\,\mathbf{k} + B_1(\cos\omega t\,\mathbf{i} + \sin\omega t\mathbf{j}). \tag{4}$$

Find also in this case the time evolution of $\mathbf{r}$.

c) Study the motion found in b) in the special case of resonance, $\omega = \omega_0 \equiv -\frac{eB_0}{m}$. Determine $\mathbf{r}(t)$ and make a qualitative description of the motion.

2 Charged particle in a strong magnetic field.

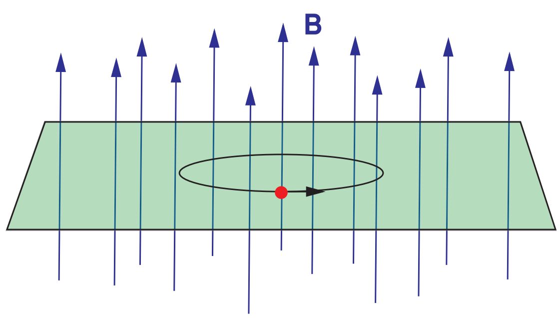

Figure 1:

We study in this problem a particle with electric charge e that moves in a strong magnetic field $\mathbf{B}$. The motion is assumed to be constrained to a plane (the x,y-plane) with the magnetic field orthogonal to the plane. The magnetic field is assumed to be constant over the plane, and eB is taken to be *negative*, with B as the z-component of $\mathbf{B}$. We assume in the following that the rotationally symmetric form of the vector potential is chosen, $\mathbf{A} = -(1/2)\mathbf{r}\times\mathbf{B}$. The relation between velocity and (canonical) momentum is $\mathbf{v} = (\mathbf{p} - e\mathbf{A})/m$, and the Hamiltonian has the standard form $H = (1/2m)(\mathbf{p} - e\mathbf{A})^2$.

We consider first the classical, non-relativistic form of the particle motion. Next we study the quantum description, where a set of coherent states is introduced for the particle in the degenerate ground state of the Hamiltonian. This description is particularly relevant for the study of the quantum Hall effect, where a 2-dimensional electron gas moves under the influence of a strong magnetic field.

a) Use Newton's second law for a charged particle in a magnetic field to show that, classically, the particle moves in a circular orbit with constant angular velocity $\omega = -eB/m$. Show, by use of the equation of motion, that generally the mechanical angular momentum $L_{mek} = m(xv_y - yv_x)$ is *not* a constant of motion, whereas $L = L_{mek} + (eB/2)r^2$ is conserved. (The last term can be viewed as an electromagnetic contribution.)

b) Consider the following vector, $\mathbf{R} = \mathbf{r} + (1/\omega)\mathbf{k}\times\mathbf{v}$, with $\mathbf{r}$ as the position and $\mathbf{v}$ as the velocity, $\mathbf{k}$ as the unit vector in the z-direction (orthogonal to the plane) and B as the z-component of the magnetic field. Show that $\mathbf{R}$ is a constant of motion, and give a physical interpretation of $\mathbf{R}$ and $\rho = (1/\omega)\mathbf{k} \times \mathbf{v}$ for the circular orbits? $\mathbf{R}$ is known as the guiding center coordinate.

c) In the quantum description, the position $\mathbf{r}$ and momentum $\mathbf{p}$ are, in the standard way replaced by operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ that satisfy the Heisenberg commutation relations. Show that the two components $\hat{X}$ and $\hat{Y}$ of the vector $\hat{\mathbf{R}}$, in the quantized form, do not commute. In what sense is the X,Y-plane similar to a two-dimensional phase space? Examine also commutators between the components $\hat{\rho}_x$ and $\hat{\rho}_y$ of $\hat{\boldsymbol{\rho}}$ in the same way.

d) Introduce dimensionless operators

$$\hat{a} = \frac{1}{\sqrt{2}l_B}(\hat{X} + i\hat{Y}) \ , \quad \hat{b} = \frac{1}{\sqrt{2}l_B}(\hat{\rho}_x - i\hat{\rho}_y) \tag{5}$$

where l_B is the so-called magnetic length, $l_B = \sqrt{\hbar/|eB|}$. Show that the set of operators $\{\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger\}$ satisfy the same commutation algebra as that of two independent harmonic oscillators. The corresponding set of harmonic oscillator states we denote by $|m, n\rangle$, where $\hat{a}^\dagger$ acts as a raising operator on the m quantum number and $\hat{b}^\dagger$ as a raising operator on the n quantum number.

e) Find expressions for the Hamiltonian $\hat{H}$ and angular momentum $\hat{L}$ in terms of the $\hat{a}$ and $\hat{b}$ operators. Show that $\hat{H}$ has an harmonic oscillator spectrum and find also the eigenvalues of $\hat{L}$ expressed in terms of m and n. In the following we assume the particle to be restricted to the degenerate ground state (the lowest Landau level). Show that this corresponds to the condition $n = 0$, while m is a free variable, so that the states $|m\rangle \equiv |m, 0\rangle, m = 0, 1, 2, ...$ form a complete set.

f) A coherent state in the Lowest Landau level can be defined by the equation

$$\hat{a}|z\rangle = z|z\rangle \ , \quad \hat{b}|z\rangle = 0 \tag{6}$$

Calculate the expectation values of the components of the position operator, $\hat{x}$ and $\hat{y}$ in the coherent state and show that it is peaked around the point $x = \sqrt{2}l_B \,\mathrm{Re}\, z, y = \sqrt{2}l_B \,\mathrm{Im}\, z$ in the x,y-plane. Use the coherent state representation for the $|m\rangle$ states, to demonstrate that the number of independent states in the lowest Landau level increases linearly with the available area in the x,y-plane. Find the density of states in the x,y-plane.

g) We assume that a weak, constant electric field E is introduced in the x-direction. Show that this effectively introduces the following Hamiltonian in the lowest Landau level,

$$H' = \frac{1}{2}\hbar\omega - \frac{l_b}{\sqrt{2}} eE(\hat{a} + \hat{a}^\dagger) \tag{7}$$

Also show that this Hamiltonian gives a time dependence to the coherent state $|z(t)\rangle$, corresponding to a drift with constant velocity in the y-direction. What is the drift velocity?