FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2007

The problem set consists of 2 problems written on 4 pages.
This set is available from Friday October 19.

Deadline for returning solutions
is Friday October 26.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471, email: j.m.leinaas@fys.uio.no).
Language
Solutions may be written in Norwegian or English, depending on your preference.

For solving the problems, it may be useful to consult the relevant sections of the lecture notes.

PROBLEMS

1 Density operators

A density operator of a two-level system can be represented by a 2×2 (density) matrix in the form

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad |\mathbf{r}| \leq 1 \tag{1}$$

where $\mathbb{1}$ is the 2×2 identity matrix, $\mathbf{r}$ is a vector in three dimensions and $\boldsymbol{\sigma}$ is a vector operator with the Pauli matrices as the Cartesian components. Geometrically the set of all density matrices form of a sphere in three dimensions, with the pure states $|\mathbf{r}| = 1$ as the surface of the sphere (the Bloch sphere), and the mixed states as the interior of the sphere.

a) The density operator can also be expressed in bra-ket formulation as

$$\hat{\rho} = \rho_{11}\,|+\rangle\langle+| \;+\; \rho_{12}\,|+\rangle\langle-| \;+\; \rho_{21}\,|-\rangle\langle+| \;+\; \rho_{22}\,|-\rangle\langle-| \tag{2}$$

where $|\pm\rangle$ is the state of the upper/lower level of the system, that is with $\sigma_z|\pm\rangle = \pm|\pm\rangle$. What are the coefficients $\rho_{ij},\ i,j = 1,2$, expressed in terms of the Cartesian components x, y, z of $\mathbf{r}$?

We consider in the following a composite system with two subsystems $\mathcal{A}$ and $\mathcal{B}$. These are both two-level systems so that the Hilbert space of the full system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is of dimension 4. A density matrix of the composite system can be written as

$$\hat{\rho} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \sum_i a_i\, \sigma_i \otimes \mathbb{1} + \sum_j b_j\, \mathbb{1} \otimes \sigma_j + \sum_{ij} c_{ij}\sigma_i \otimes \sigma_j) \tag{3}$$

with a_i, b_j and c_{ij} as coefficients, and with the first factor in the tensor product corresponding to the $\mathcal{A}$ subsystem and the other to $\mathcal{B}$.

b) Find the reduced density matrices of subsystems $\mathcal{A}$ and $\mathcal{B}$ expressed in terms of the a, b and c coefficients. What condition should the a, b and c coefficients satisfy if the two subsystems should be completely uncorrelated?

We examine the four *Bell states* of the composite system,

$$\begin{aligned} |c, \pm\rangle &= \tfrac{1}{\sqrt{2}}\,(|++\rangle \pm |--\rangle) \\ |a, \pm\rangle &= \tfrac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle) \end{aligned} \tag{4}$$

where $|ij\rangle = |i\rangle \otimes |j\rangle$, $i, j = \pm$, are tensor product states.

c) Give the expressions for the density operators of the four states, first in the bra-ket form, and then written in the form (3). What are the reduced density matrices of subsystems $\mathcal{A}$ and $\mathcal{B}$ for these four states? Give the entropy of the full system and the two subsystems in the four cases. Why do we call the Bell states *maximally entangled*?

d) We consider linear combinations of the form

$$\hat{\rho} = x\hat{\rho}_1 + (1-x)\hat{\rho}_2 \tag{5}$$

where $\hat{\rho}_1$ and ρ_2 represent two Bell states and x is a real parameter. Show that if we have $0 < x < 1$ the linear combination is a density operator. Why is that not the case if $x < 0$ or $x > 1$?

e) Choose a pair of Bell states and show that halfway between them ($x = 1/2$) the density matrix gets a particularly simple form. Show that it can be written in the form

$$\hat{\rho} = \frac{1}{8}\left[(\mathbb{1} + \mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} + \mathbf{m}\cdot\boldsymbol{\sigma}) + (\mathbb{1} - \mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} - \mathbf{m}\cdot\boldsymbol{\sigma})\right] \tag{6}$$

where $\mathbf{n}$ is a unit vector and $\mathbf{m} = \pm\mathbf{n}$. What does this expression show about entanglement between the two subsystems $\mathcal{A}$ and $\mathcal{B}$ for this particular state?

f) The Bell states define a subspace in the space of all 4×4 density matrices. Show that the density matrices in this subspace commute.

2 Jaynes-Cummings model

The Jaynes-Cummings model is a simplified model for the system of an atom interacting with the electromagnetic field. One assumes that only two of the atomic energy levels are involved in the interaction, so that the atom can be modelled as a two-level system. One further assumes that only one field mode is excited, so that the field can be modelled by an harmonic oscillator. In this oscillator model the different energy levels correspond to different numbers of photons in the excited field mode. The situation is most relevant for an atom in a reflecting cavity where a single mode of the field can be strongly excited.

We write the Hamiltonian of the model in the following way

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + i\hbar\lambda(a^\dagger\sigma_- - a\sigma_+) \equiv \hat{H}_0 + \hat{H}_1 \tag{7}$$

where the $\hat{H}_0$ includes the two first terms, which describe the non-interacting atom and photons, and $\hat{H}_1$ the third term, which describes interactions between the atoms and the photons. The expression $\hbar\omega_0$ in the first term gives the energy difference between the two atomic levels, while $\hbar\omega$ is the photon energy. (The zero point of the energy has been adjusted to absorb the ground state energy of the harmonic oscillator and to place the energies of the two-level system symmetrically about $E = 0$.) The Pauli matrices act as operators between the atomic levels, with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$, and $a^\dagger$ and a are operators that create and destroys a photon. The interaction term thus has two parts, where the first part creates a photon while lowering the atomic energy and the other part destroys a photon while increasing the atomic energy. λ is a real valued parameter that determines the strength of the interaction. The simple form of the interaction $\hat{H}_1$ given here is valid in the *rotating wave approximation*. This gives a good approximation to the full interaction when the two frequencies ω_0 and ω are close in value.

The objective is to study the time evolution of this system, where the interaction term will induce oscillations between the atomic levels. These oscillations are called *Rabi oscillations* and are examined in a somewhat different way in Sect. 1.3.2 of the lecture notes. There the electromagnetic field was treated as an external time dependent perturbation, while we here include the field as a part of the full quantum system and describe it in terms of photons.

We use the notation $|m, n\rangle$ for the eigenstates of the non-interacting Hamiltonian $\hat{H}_0$, with $m = \pm 1$ indicating the atomic state and $n = 0, 1, 2, ...$ indicating the number of photons (which is here the level number of the harmonic oscillator).

a) Show that the interaction Hamiltonian $\hat{H}_1$ couples the unperturbed levels only in pairs that differ by one photon. We define such a pair of states as $|1\rangle \equiv |1, n-1\rangle$ and $|2\rangle \equiv |-1, n\rangle$. Show that the Hamiltonian in the subspace spanned by this pair of states can be written as a 2x2 matrix of the form

$$H = \frac{1}{2}\hbar \begin{pmatrix} \Delta & -ig \\ ig & -\Delta \end{pmatrix} + \epsilon\mathbb{1} \tag{8}$$

with $\mathbb{1}$ as the 2×2 identity matrix, and find the expressions for Δ, g and ϵ.

b) Solve the eigenvalue problem for this 2x2 matrix Hamiltonian and find the energy eigenvalues and the eigenvectors in matrix form. To simplify expressions it may be convenient to write the matrix elements in terms of new parameters Ω and θ defined by

$$\Delta = \Omega\cos\theta\,, \quad g = \Omega\sin\theta\,, \tag{9}$$

c) In matrix form a general time dependent state can be written as

$$\psi(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \tag{10}$$

Find the time dependent coefficients $c_1(t)$ and $c_2(t)$ expressed in terms of Ω and θ for the initial condition $c_1 = 0$, $c_2 = 1$ at time $t = 0$. Show that $|c_1(t)|^2 = \sin^2\theta \sin^2\frac{\Omega t}{2}$.

d) Give a qualitative description of the result for the excitation of the atom, and make a comparison with the result of Sect. 1.3.2 of the lecture notes. How does the field strength B_1 in the lecture notes relate to the photon number n in the present case?

The system consisting of the atom and the photons can be considered as a composite quantum system, where the atom is subsystem $\mathcal{A}$ and the electromagnetic field (the photons) defines subsystem $\mathcal{B}$. The Hilbert space of the full system is then a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The eigenstates of $\hat{H}_0$ referred to above are special cases of tensor product states,

$$\begin{aligned} |1\rangle &= |1, n-1\rangle = |+1\rangle_A \otimes |n-1\rangle_B \\ |2\rangle &= |-1, n\rangle \;\; = |-1\rangle_A \otimes |n\rangle_B \end{aligned} \tag{11}$$

e) Write the time dependent state (10) as a "ket" vector expanded in the above product states, and give the expression for the corresponding density operator in the bra-ket form. (Write the expressions in terms of $c_1(t)$ and $c_2(t)$ without using the solutions for these.)

f) Show that the reduced density matrix of the atom (subsystem $\mathcal{A}$) can be written as a 2x2 matrix that depends only on $|c_1|^2$ and $|c_2|^2$. Find the corresponding von Neumann entropy expressed in terms of θ and Ω and plot this as a time dependent function for several different values of $\sin\theta$ with Ω fixed. What do these plots show about variations in the entanglement between the atom and the electromagnetic field?