UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 4110/ 9110 Non-relativistic quantum mechanics
Day of exam: Wednesday, December 7, 2011
Exam hours: 4 hours, beginning at 14:30
This examination paper consists of 2 problems on 3 pages
Permitted materials: Calculator
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before you begin.

PROBLEM 1
Dressed photon states

A photon is interacting with an atom within a small reflecting cavity. The photon energy is close to the excitation energy of the atom, which connects the ground state to the first excited state. Due to interaction between the photon and the atom the stationary states of the composite system are admixtures of the photon and atomic states. These are sometimes referred to as a ”dressed” photon states. In this problem we examine some of the properties of the dressed states.

The Hamiltonian of the photon-atom system can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \tag{1}$$

where $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The Pauli matrices act between the two atomic levels, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. $\hat{a}$ and $\hat{a}^\dagger$ are the photon creation and destruction operators.

a) We introduce the notation $|+,0\rangle = |+\rangle \otimes |0\rangle$ and $|-,1\rangle = |-\rangle \otimes |1\rangle$ for the relevant product states of the composite system, with $0, 1$ referring to the photon number. Show that in the two-dimensional subspace spanned by these vectors the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1} \tag{2}$$

where we assume $|-,1\rangle$ to correspond to the lower matrix position and $|+,0\rangle$ to the upper one. $\mathbb{1}$ denotes the 2×2 identity matrix. Express the parameters Δ, $\cos\phi$, $\sin\phi$, and ϵ in terms of ω_0, ω and λ.

b) Find the energy eigenvalues $E_\pm$. Find also the eigenstates $|\psi_\pm(\phi)\rangle$, expressed in terms of the product states $|+,0\rangle$ and $|-,1\rangle$, and show that they are related by $|\psi_-(\phi)\rangle = |\psi_+(\phi+\pi)\rangle$.

In the following we focus on the state $|\psi_-(\phi)\rangle$, which we assume to be the one-photon state in the non-interacting case. This state (with a convenient choice of phase factor) can be written as $|\psi_-(\phi)\rangle = \cos\frac{\phi}{2}\,|-,1\rangle + i\sin\frac{\phi}{2}\,|+,0\rangle$.

c) Find expressions for the reduced density operators of the photon and of the atom for the state $|\psi_-(\phi)\rangle$. Discuss in what parameter interval the state is mostly a photon-like state and when it is mostly an atom-like state.

d) Determine the entanglement entropy as a function of ϕ, and find for what values the entropy is minimal and maximal. Relate this to the discussion in c).

e) At time $t = 0$ a single photon is sent into the cavity, where there is previously no photon and the atom is in its ground state. Determine the time dependent probability $p(t)$ for a photon later to be present in the cavity. Give a qualitative explanation of its oscillatory behaviour, and specify what determines the frequency and amplitude of the oscillations.

PROBLEM 2
A radiation problem

We consider a one-dimensional problem where a two-level system (A) interacts with a scalar radiation field (B). The notation we use is essentially the same as in Problem 1. The Hamiltonian of the system we consider is

$$\hat{H} = \frac{1}{2}\hbar\omega_A\,\sigma_z + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \kappa\sum_k \sqrt{\frac{\hbar}{2L\omega_k}}(\hat{a}_k\sigma_+ + \hat{a}_k^\dagger\sigma_-) = \hat{H}_0 + \hat{H}_{int} \tag{3}$$

The first term is the two-level Hamiltonian, with energy splitting $\hbar\omega_A$, the second one is the free field contribution, with $k = 2\pi n/L$ (n - integer) as the wave number of the photon. L is a (large) normalization length. The third term is the interaction term $\hat{H}_{int}$, with κ as an interaction parameter. The frequency parameter is $\omega_k = ck$.

a) A general state of the two-level system is characterized by a vector $\mathbf{r}$, with $r \leq 1$, and with the corresponding density matrix as

$$\rho_A = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{4}$$

Consider first that the interaction term $\hat{H}_{int}$ is turned off, $\kappa = 0$, so that the time evolution operator of the two-level system is $\hat{\mathcal{U}}(t) = \exp(-\frac{i}{2}\omega_A\,t\,\sigma_z)$. Use this to determine the the density matrix $\rho_A(t)$ at time t, assuming that $\rho_A(0)$ is identical to the density matrix in (4), and show that the time evolution of $\mathbf{r}$ is a precession around the z-axis with angular velocity ω_A.

b) Assume next that $\kappa \neq 0$ and that initially the two-level system is in the excited "spin up state", while the scalar field is in the vacuum state. Thus, the initial state is $|+,0\rangle = |+\rangle\otimes|0\rangle$. It decays to the "spin down state" by emission of a field quantum. The final state we then write as $|-,1_k\rangle = |-\rangle\otimes|1_k\rangle$.

The occupation probability of the excited state $|+\rangle$ decays exponentially, $P_+(t) = \exp(-\gamma t)$, with a decay rate γ that to first order in the interaction, and in the limit $L\to\infty$, is given by

$$\gamma = \frac{L}{(2\pi\hbar)^2}\int dk\,|\langle -,1_k|\hat{H}_{int}|+,0\rangle|^2\,\delta(\omega_k - \omega_A) \tag{5}$$

Determine the decay rate γ, expressed in terms of the parameters of the problem.

As discussed in the lectures an approximate way to handle the decay is to introduce an imaginary contribution to the energy of the decaying state. Assuming a more general initial state, of the form

$$|\psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle) \otimes |0\rangle = \alpha|+,0\rangle + \beta|-,0\rangle \tag{6}$$

with α and β as unspecified coefficients, with $|\alpha|^2 + |\beta|^2 = 1$, we make the corresponding *ansatz* for the time evolved state

$$|\psi(t)\rangle = (e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle) \otimes |0\rangle + \sum_k c_k(t)|-,1_k\rangle \tag{7}$$

with $c_k(t)$ as decay parameters, which satify $c_k(0) = 0$.

c) Check what normalization of the state vector (7) means for the decay parameters, and determine the reduced density matrix matrix $\rho_A(t)$ of the two-level system.

d) Assume the same initial conditions as in b), $z(0) = 1$, $x(0) = y(0) = 0$ ($\alpha = 1$, $\beta = 0$). Determine the density matrix $\rho_A(t)$ and the corresponding time dependent vector $\mathbf{r}(t)$. Is the time evolution consistent with the expected exponential decay of the excited state of the two-level system? Give a brief description of the evolution of the entanglement between the two level system and the radiation field during the decay.

e) Choose another initial condition $x(0) = 1$, $y(0) = z(0) = 0$ ($\alpha = \beta = 1/\sqrt{2}$), and find also in this case the time evolution of the reduced density matrix and the components of the vector $\mathbf{r}(t)$. Sketch the time evolution of $\mathbf{r}(t)$ and compare qualitatively the motion with that in a) and d). Find $r(t)^2$ expressed as a function of γt, and sketch also this function. What does it show about the time evolution of the entanglement between the two subsystems A and B?

Assume in this paragraph $\gamma << \omega_A$.