UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS4110/9110 Modern Quantum Mechanics
Day of exam: Tuesday, December 8, 2015
Exam hours: 4 hours, beginning at 14:30
This examination paper consists of 3 problems, written on 3 pages
Permitted materials: Approved calculator
Angell and Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference. Noen engelske ord er oversatt etter hver oppgave.

Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1
Two spin-half systems

A quantum system is composed of two interacting spin-half systems, referred to as system A and B. The Hamiltonian has the form

$$\hat{H} = \frac{1}{2}\hbar\omega(\,\sigma_z \otimes \mathbb{1} - \,\mathbb{1} \otimes \sigma_z) + \hbar\lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

where σ_z og $\sigma_\pm$ are Pauli matrices, with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hbar\omega$ giving the splitting between the two energy levels of each of the two spins, and with λ as a coupling parameter. The two factors of the tensor product refer to each of the two spin systems, with A corresponding to the first and B as the second factor. It is convenient to introduce new parameters by $\omega = a\cos\theta$ and $\lambda = a\sin\theta$, with $-\pi/2 < \theta \leq \pi/2$. We further use $|\pm\rangle$ as notation for the eigenstates of σ_z. In the following we use the tensor products of these states as basis for the Hilbert space of the composite system.

a) Show that only the product states $|+-\rangle = |+\rangle \otimes |-\rangle$ and $|-+\rangle = |+\rangle \otimes |-\rangle$ are mixed by the λ term in the Hamiltonian, and give the expression for the Hamiltonian as a 2x2 matrix, in the subspace spanned by $|+-\rangle$ and $|-+\rangle$.

b) Find the energy eigenvalues, and the energy eigenstates, expressed in terms of a and θ.

c) Determine, for the energy eigenstates, the density operator of the full system and the reduced density operators of the two subsystems, and detemine the entanglement entropy of the eigenstates as functions of θ. What are the minimum and maximum values of the entropy functions? Make a comparison with the maximal possible value of the entanglement entropy of the two-spin system.

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density operator = tetthetsoperator
entanglement = sammenfiltring

PROBLEM 2
A driven harmonic oscillator

A quantum mechanical, driven harmonic oscillator is described by the following Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \tag{2}$$

where $\hat{a}$ og $\hat{a}^\dagger$ satisfy the standard commutation relations for lowering and raising operators, and where ω_0, ω og λ are three constants.

a) As a reminder, Heisenberg's equation of motion has the form

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar}\left[H, \hat{A}\right] + \frac{\partial}{\partial t}\hat{A} \tag{3}$$

for any given observable $\hat{A}$. Apply this to the operator $\hat{a}_H$, which is the operator $\hat{a}$ transformed to the Heisenberg picture, and show that it satisfies an equation of the form

$$\frac{d^2\hat{a}_H}{dt^2} + \omega_0^2\hat{a}_H = Ce^{-i\omega t}\,\mathbb{1} \tag{4}$$

with C as a constant. Determine C.

b) Equation (4) can be solved as a linear differential equation, to give

$$\hat{a}_H(t) = \hat{a}\,e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\,\mathbb{1} \tag{5}$$

Show that (5) is a solution of (4) and determine the constant D.

c) A coherent state is defined as an eigenstate of the lowering operator $\hat{a}$,

$$\hat{a}|z\rangle = z|z\rangle \tag{6}$$

Assume that the oscillator, at time $t = 0$, is in the ground state for the λ-independent part of the Hamiltonian, that is

$$|\psi(0)\rangle = |0\rangle\,, \qquad \hat{a}|0\rangle = 0 \tag{7}$$

Show that, during the time evolution (in the Schrödinger picture), it will continue as a coherent state, so that

$$\hat{a}|\psi(t)\rangle = z(t)|\psi(t)\rangle \tag{8}$$

with $z(t)$ as a complex-valued function of time.

Find the function $z(t)$, and compare the time evolution of the real part $x(t) = (z(t)+z(t)^*)/2$ with the motion of the corresponding classical driven harmonic oscillator.

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driven harmonic oscillator = tvungen harmonisk oscillator
coherent state = koherent tilstand

PROBLEM 3
Atom and photon in an optical microcavity

An atom is contained in an optical microcavity, with the energy difference between two of the atomic levels matching exactly the frequency of one of the electromagnetic cavity modes. A simplified description of the photon-atom system has the form of a two-level system coupled to a single electromagnetic mode. The Hamiltonian then takes the form

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{9}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are photon creation and annihilation operators, and σ_z and $\sigma_\pm$ are Pauli matrices with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. These operators act between the two atomic levels with the upper and lower energy levels corresponding to the eigenvalues $+1$ and -1 respectively of σ_z. We refer in the following to $|\pm, n\rangle = |\pm\rangle \otimes |n\rangle$ as product states of the composite system, with $|\pm\rangle$ as the upper/lower atomic levels and $|n\rangle$ as the photon number states of the cavity mode.

a) Assume a single photon is introduced in the cavity at time $t = 0$ while the atom is in its ground state. Show that the atom-photon state will subsequently oscillate in the following way

$$|\psi(t)\rangle = e^{i\epsilon t}(\cos\Omega t\,|-,1\rangle - i\sin(\Omega t)|+,0\rangle) \tag{10}$$

and find Ω and ϵ expressed in terms of ω and λ.

To take into account leakage of photons from the cavity, we turn to a description of the time evolution in terms of the density operator. It is assumed to satisfy the Lindblad equation,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H, \hat{\rho}] - \frac{1}{2}\gamma\left[\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right] \tag{11}$$

where γ is the escape rate for photons from the cavity.

b) The probability for finding the atom in the ground state with no photon in the cavity is $p_g = \langle -,0|\hat{\rho}|-,0\rangle$. Assume that there is initially a non-vanishing probability for a photon being present in the cavity. Show that that this will result in an increase in p_g with time, which is consistent with the expectation that the photon will escape from the cavity.

c) Assuming there is no contribution to $\hat{\rho}$ from higher excited states than $|-,1\rangle$ and $|+,0\rangle$, show that a closed set of coupled differential equations for the three variables $p_1 = \langle -,1|\hat{\rho}|-,1\rangle$, $p_0 = \langle +,0|\hat{\rho}|+,0\rangle$ and $b = \mathrm{Im}\langle -,1|\hat{\rho}|+,0\rangle$ can be derived from the Lindblad equation.

Without solving the equations, give a qualitative description of what the expected time evolution will be with the same initial condition as in a).

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cavity mode = kavitetsmode, hulromsmode