UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
Day of exam:	**8. December 2020**
Exam hours:	**9.00-13.00, 4 hours**
This examination paper consists of 4 pages	
Permitted materials:	**Any written material (including internet). You may not collaborate with other students or ask any other person.**

Language: The solutions may be written in Norwegian or English depending on your own preference.
Make sure that your copy of this examination paper is complete before answering.
All answers should be justified

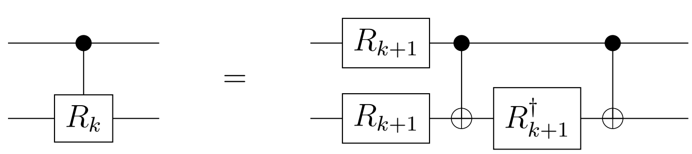
PROBLEM 1

Quantum circuit for controlled R_k

a) In the quantum Fourier transformation, we needed to perform a controlled R_k operation. The one-qubit operator

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

is then performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. We know that all two-qubit operators can be decomposed in single qubit operators and controlled NOT (CNOT) operations. Show that the following quantum circuit is one such decompostion for the controlled R_k operation



b) We consider now general controlled U operations, with U a one-qubit operator. This means that the operation U is performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. In both cases, the control qubit is not changed. If this was a classical system, this would be all the possibilities, but in a quantum system, one can have a control qubit that is in a superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ of the two basis states. In general, the two qubits will be entangled by this operation, so no definite quantum state can be ascribed to any of them. However, a special situation arises if the initial state of the target qubit is an eigenstate of U. Draw a quantum circuit desribing this situation. Show that in this case, the two qubits are not entangled by the operation. Show also that in this case, it is the target qubit that is not changed, while the state of the control qubit is changed. Find the final state of the control qubit in terms of the eigenvalues of U.

c) This result is surprising if we only are used to the classical world, and deserves an explanation. Explain in words why the target is not changed while the state of the control does change.

PROBLEM 2

Destruction of entanglement by noise

We have two two-level systems, A and B. Each system has a basis for its Hilbertspace with vector representation

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We introduce a vector representation of the tensor product as described in Problem 5.3 from the exercises. Assume that the density matrix for the joint system is of the form

$$\rho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{1}$$

a) Determine for which values of the parameters a, b, c, d and z this represents a pure state for the joint system.

b) Find the reduced density matrices for systems A and B. In those cases where ρ represents a pure state, determine if the state is entangled or not.

We now specify the Hamiltonian for the two two-level systems as

$$H = \frac{1}{2}\hbar\omega\sigma_z^A + \frac{1}{2}\hbar\omega\sigma_z^B.$$

where $\sigma_z^A = \sigma_z \otimes \mathbb{1}$ and $\sigma_z^B = \mathbb{1} \otimes \sigma_z$. The system is in contact with an environment which means that the density matrix is evolving according to the Lindblad equation

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}\left[\sigma_+^A\sigma_-^A\rho + \rho\sigma_+^A\sigma_-^A - 2\sigma_-^A\rho\sigma_+^A\right] - \frac{\gamma}{2}\left[\sigma_+^B\sigma_-^B\rho + \rho\sigma_+^B\sigma_-^B - 2\sigma_-^B\rho\sigma_+^B\right].$$

c) What is the temperature of the environment described by this Lindblad equation? Justify your answer.

One can show that if the density matrix at time $t = 0$ is of the form (1) it will have this form at all later times, with time dependent matrix elements $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $z(t)$. If we call the initial values of these variables a_0, b_0, c_0, d_0 and z_0, the solution of the Lindblad equation is

$$\begin{aligned}
a(t) &= a_0 e^{-2\gamma t} \\
b(t) &= b_0 e^{-\gamma t} + a_0 e^{-\gamma t}(1 - e^{-\gamma t}) \\
c(t) &= c_0 e^{-\gamma t} + a_0 e^{-\gamma t}(1 - e^{-\gamma t}) \\
d(t) &= 1 - (b_0 + c_0) e^{-\gamma t} - a_0 e^{-\gamma t}(2 - e^{-\gamma t}) \\
z(t) &= z_0 e^{-\gamma t}
\end{aligned} \tag{2}$$

You do not have to show this, but can use it in the following.

d) Assume the initial conditions

$$a_0 = 1, \qquad b_0 = c_0 = d_0 = z_0 = 0.$$

Find the von Neumann entropy of the state as a function of time. Plot/sketch the entropy as a function of time, and comment on the form of the function.

We have seen that when the full system is in a pure state, we can measure entanglement by the entanglement entropy. If the full system is a mixed state this is not a good measure of entanglement.

e) Give an example of a separable state of two systems where the entropy of entanglement is large.

To study the evolution of the entanglement in our system, we need to quantify the entanglement for the situation where the full system is not in a pure state. One common measure of entanglement is the concurrence. To calculate it we defince the matrix

$$M = \rho \sigma_y^A \otimes \sigma_y^B \rho^* \sigma_y^A \otimes \sigma_y^B$$

where ρ^* is the elementvise complex conjugate of ρ. The concurrence is defined as

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

where λ_i are the square roots of the eigenvalues of M sorted in descending order (λ_1 is the largest, λ_4 is the smallest).

f) Show that the concurrence as a function of time for the density matrix (1) with the elements given by the solution (2) with the initial conditions $d_0 = \frac{1}{3} - a_0$, $b_0 = c_0 = z_0 = \frac{1}{3}$ is

$$C = \max\left(0, \frac{2}{3}e^{-\gamma t} - 2e^{-\gamma t}\sqrt{a_0}\sqrt{1 - \frac{2}{3}e^{-\gamma t} - a_0 e^{-\gamma t}(2 - e^{-\gamma t})}\right).$$

g) Assume now that $a_0 = \frac{1}{3}$. Show that the concurrence goes to 0 in a finite time, and find this time.