FYS 4110/9110 Modern Quantum Mechanics
Exam, Fall Semester 2020. Solution

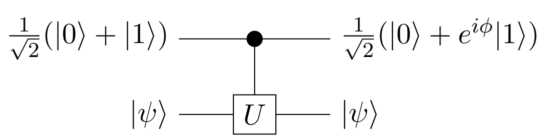
Problem 1: Quantum circuit for controlled R_k

a) We define $\phi = 2\pi/2^k$ and get

$$\begin{aligned}
|\psi_1\rangle \otimes |\psi_2\rangle &= (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle) \\
&\stackrel{R_{k+1}}{\rightarrow} (a_0|0\rangle + a_1 e^{i\phi/2}|1\rangle) \otimes (b_0|0\rangle + b_1 e^{i\phi/2}|1\rangle) \\
&\stackrel{CNOT}{\rightarrow} a_0|0\rangle \otimes (b_0|0\rangle + b_1 e^{i\phi/2}|1\rangle) + a_1 e^{i\phi/2}|1\rangle \otimes (b_0|1\rangle + b_1 e^{i\phi/2}|0\rangle) \\
&\stackrel{R^\dagger_{k+1}}{\rightarrow} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1 e^{i\phi/2}|1\rangle \otimes (b_0 e^{-i\phi/2}|1\rangle + b_1 e^{i\phi/2}|0\rangle) \\
&\stackrel{CNOT}{\rightarrow} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1|1\rangle \otimes (b_0|0\rangle + b_1 e^{i\phi}|1\rangle) \\
&= a_0|0\rangle \otimes |\psi_2\rangle + a_1|1\rangle \otimes R_k|\psi_2\rangle
\end{aligned}$$

This is the controlled R_k operation.

b) Let $U|\psi\rangle = e^{i\phi}|\psi\rangle$. The situation is described by this circuit



The evolution of the state is

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \otimes |\psi\rangle &\stackrel{\text{control}-U}{\rightarrow} \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle) \otimes \psi.
\end{aligned}$$

c) Since multiplying by a phase factor does not change a quantum state, U does not really change the state of the target if the initial state is an eigenstate. However, the relative phase between two states does make a physical difference. Therefore, when the control is in a superposition, there is a phase difference between the two states after the control-U operation. Since the state of the target is the same in both cases, it factors out, leaving a product state with the relative phase between the two states of the control qubit.

Problem 2: Destruction of entanglement by noise

a) ρ is a pure state if one eigenvalue is 1 and the rest 0.

$$\begin{vmatrix} a-\lambda & 0 & 0 & 0 \\ 0 & b-\lambda & z & 0 \\ 0 & z^* & c-\lambda & 0 \\ 0 & 0 & 0 & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda)[(b-\lambda)(c-\lambda) - |z|^2] = 0$$

which gives the eigenvalues

$$\lambda_a = a, \qquad \lambda_d = d, \qquad \lambda_\pm = \frac{1}{2}(b+c) \pm \sqrt{\frac{1}{4}(b-c)^2 + |z|^2}. \tag{1}$$

Thus we have that ρ is pure if

1: $a = 1, b = c = d = z = 0$.

2: $b = 1, a = b = c = z = 0$.

3: $a = d = 0$. Since $\mathrm{Tr}\,\rho = 1$ we must then have $b + c = 1$. This means that

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4}(b-c)^2 + |z|^2}.$$

For ρ to be pure we must have $\lambda_+ = 1$ and $\lambda_- = 0$, and therefore

$$\frac{1}{4}(b-c)^2 + |z|^2 = \frac{1}{4}$$

which gives

$$|z|^2 = \frac{1}{4}[1 - (b-c)^2] = \frac{1}{4}[1 - (2b-1)^2]$$

where we used that $c = 1 - b$. Since $|z|^2 > 0$, b is restricted to the interval $0 \leq b \leq 1$.

b) We write ρ on the form

$$\rho = a|11\rangle\langle 11| + b|10\rangle\langle 10| + c|01\rangle\langle 01| + d|00\rangle\langle 00| + z|10\rangle\langle 01| + z^*|01\rangle\langle 10|$$

from which we read out

$$\rho^A = \mathrm{Tr}_B\,\rho = (a+b)|1\rangle\langle 1| + (c+d)|0\rangle\langle 0| = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix},$$

$$\rho^B = \mathrm{Tr}_A\,\rho = (a+c)|1\rangle\langle 1| + (b+d)|0\rangle\langle 0| = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix}.$$

We check the three cases of pure ρ from question a)

1: $a = 1, b = c = d = z = 0$:

$$\rho^A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho^B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is not entangled since ρ^A and ρ^B are pure.

2: $d = 1, a = b = c = z = 0$: By symmetry with case 1, this is not entangled.

3: $a = d = 0, 0 \leq b \leq 1, c = 1 - b, |z|^2 = \frac{1}{4}[1 - (2b - 1)^2]$:

$$\rho^A = \begin{pmatrix} b & 0 \\ 0 & 1-b \end{pmatrix}, \qquad \rho^B = \begin{pmatrix} 1-b & 0 \\ 0 & b \end{pmatrix}.$$

This is entangled for all $b \neq 0, 1$.

c) The two Lindbladoperators are σ_-^A and σ_-^B. Both correspond to transitions $|1_{A/B}\rangle \to |0_{A/B}\rangle$ that reduce the energy (we assume $\omega > 0$), emitting energy to the environment. This means that the environment is at $T = 0$.

d) With the given initial conditions, the matrix elements are

$$a(t) = e^{-2\gamma t}, \qquad b(t) = c(t) = e^{-\gamma t}(1 - e^{-\gamma t}), \qquad d(t) = (1 - e^{-\gamma t})^2, \qquad z(t) = 0.$$

The von Neumann entropy is given as

$$S = -\operatorname{Tr} \rho \ln \rho = -\sum_i \lambda_i \ln \lambda_i$$

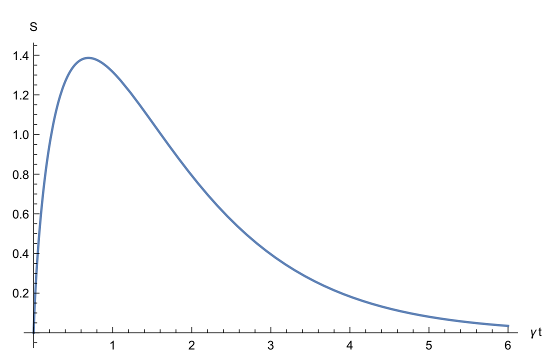
where λ_i are the eigenvalues of ρ. Using (1) we get

$$\lambda_a = e^{-2\gamma t}, \qquad \lambda_d = (1 - e^{-\gamma t})^2, \qquad \lambda_\pm = e^{-\gamma t}(1 - e^{-\gamma t})$$

The entropy is then

$$S = -e^{-2\gamma t} \ln e^{-2\gamma t} - (1-e^{-\gamma t})^2 \ln(1-e^{-\gamma t})^2 - 2e^{-\gamma t}(1-e^{-\gamma t}) \ln[e^{-\gamma t}(1-e^{-\gamma t})] = 2\gamma t - 2(1-e^{-\gamma t}) \ln(e^{\gamma t}-1).$$

We plot $S(t)$



We see that the entropy is zero at $t = 0$, corresponding to the initial state being pure. As time increases, the system goes to a mixed state and the entropy increases. Since $T = 0$, the system will approach the ground state, and the entropy decreases again, approaching zero at $t \to \infty$.

e)

$$\rho = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$S = \ln 2$ which is maximal for two-level systems.

f) We need to find

$$\sigma_y^A \otimes \sigma_y^B = \begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}$$

and calculate

$$M = \rho\sigma_y^A \otimes \sigma_y^B \rho^* \sigma_y^A \otimes \sigma_y^B = \begin{pmatrix}ad & 0 & 0 & 0\\ 0 & bc + |z|^2 & 2bz & 0\\ 0 & 2cz^* & bc + |z|^2 & 0\\ 0 & 0 & 0 & ad\end{pmatrix}.$$

Two of the eigenvalues of M are

$$\mu_a = \mu_d = ad.$$

The other two we find from

$$\begin{vmatrix}bc + |z|^2 - \mu & 2bz\\ 2cz^* & bc + |z|^2 - \mu\end{vmatrix} = (bc + |z|^2 - \mu)^2 - 4bc|z|^2 = 0$$

which gives

$$\mu_\pm = (\sqrt{bc} \pm |z|)^2.$$

With the initial conditions $d_0 = \frac{1}{3} - a_0$, $b_0 = c_0 = z_0 = \frac{1}{3}$ we get

$$\sqrt{\mu_a} = \sqrt{\mu_d} = \sqrt{ad} = e^{-\gamma t}\sqrt{a_0}\sqrt{1 - \frac{2}{3}e^{-\gamma t} - a_0 e^{-\gamma t}(2 - e^{-\gamma t})},$$

$$\sqrt{\mu_+} = \frac{2}{3}e^{-\gamma t} + a_0 e^{-\gamma t}(1 - e^{-\gamma t}), \qquad \sqrt{\mu_-} = a_0 e^{-\gamma t}(1 - e^{-\gamma t}).$$

The largest eigenvalue is μ_+, so $\lambda_1 = \sqrt{\mu_+}$. This gives

$$\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 = \frac{2}{3}e^{-\gamma t} - 2e^{-\gamma t}\sqrt{a_0}\sqrt{1 - \frac{2}{3}e^{-\gamma t} - a_0 e^{-\gamma t}(2 - e^{-\gamma t})}.$$

g) $C = 0$ when

$$\frac{2}{3}e^{-\gamma t} - 2e^{-\gamma t}\sqrt{a_0}\sqrt{1 - \frac{2}{3}e^{-\gamma t} - a_0 e^{-\gamma t}(2 - e^{-\gamma t})} = 0$$

which we solve to get

$$e^{-\gamma t} = \frac{1}{3a_0} + 1 \pm \frac{1}{a_0}\sqrt{a_0^2 - \frac{4}{3}a_0 + \frac{2}{9}}.$$

For $a_0 = \frac{1}{3}$ we get $e^{-\gamma t} = 2 \pm \sqrt{2}$. Since $e^{-\gamma t} < 1$ for positive t and γ, we must choose $e^{-\gamma t} = 2 - \sqrt{2}$, which means

$$t = \frac{1}{\gamma} \ln \frac{2 + \sqrt{2}}{2}.$$

At this time, the concurrence drops to exactly 0. It means that even if the state approaches the ground state asymptotically, the entanglement (as measured by the concurrence) vanishes completely in a finite time.