FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2020

Return of solutions:
The problem set is available from Friday morning, 16 October.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 23 October, at 12:00.

Language:

Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems:

Please ask Joakim Bergli (room V405).

The problem set consists of 2 problems written on 5 pages.

Problem 1: Bloch sphere for three-level system

For a two-level system with density matrix ρ we have defined the Bloch vector $\mathbf{m}$ by the equation

$$\rho = \frac{1}{2}(\mathbb{1} + m_i \sigma_i)$$

where σ_i are the Pauli matrices, and we are summing over the repeated index i. We want to generalize this to arbitrary n-level systems, and in particular study the three-level case. The density matrix is in general a Hermitian matrix with $\mathrm{Tr}(\rho) = 1$, which means that we can write

$$\rho = \frac{1}{n}(\mathbb{1} + \alpha m_i \lambda_i)$$

where α is a numerical constant that will depend on n, and λ_i are traceless Hermitian matrices.

a) How many matrices λ_i do we need for an n-level system? This will also be the number of components of the Bloch vector. That is, the number of dimensions of the space where the Bloch vector is.

One can always choose the matrices λ_i to satisfy the relation

$$Tr(\lambda_i \lambda_j) = 2\delta_{ij}$$

(see e. g. G. Kimura, Physics Letters A **314**, 339 (2003) for an explicit form).

b) Find the value of α so that pure states have $|\mathbf{m}| = 1$.

c) What is the dimension of the space of pure states for n-level systems?

d) Explain why the pure states are on the surface of the Bloch sphere, but do not cover it.

We will now specialize to the case of a 3-level system. In this case the matrices λ_i are known as the Gell-Mann matrices (for the form of all these and more on the present problem, see S. Goyal *et al.* J. Phys. A: Math. Theor. **49**, 165203 (2016)). In question d) you showed that the pure states do not cover the entire surface of the Bloch sphere. We will now see what happens for mixed states. In addition to being Hermitian and having trace 1, the density matrix should not have negative eigenvalues. We will restrict the Bloch vector to lie in the plane spanned by the two Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

so that the density matrix is of the form

$$\rho = \frac{1}{3}\left[\mathbb{1} + \sqrt{3}(m_1\lambda_1 + m_8\lambda_8)\right].$$

e) Find the eigenvalues of density matrices ρ of this form.

f) Plot the cross section of the Bloch sphere spanned by the Gell-Mann matrices λ_1 and λ_8 and mark the area where the density matrix has only positive eigenvalues. These are the only density matrices allowed as physical states.

g) Plot the von Neumann entropy for states in this plane, and determine if the entropy depends only on the length of the Bloch vector, as for a two-level system, or is also a function of the direction of the Bloch vector.

Problem 2: Entanglement transformations using local operations and classical communication

We consider a bipartite system, with subsystems A and B. If we have two pure states, generally both entangled, we can wonder if the entanglement is in some sense equivalent in the two states. By "equivalent" we do not mean quantitatively equal (that is, with the same entanglement entropy), but rather qualitatively equal (but maybe to different degree, so that the entanglement entropy could be different). One way to approach this is to study if one state can be converted to the other if we only apply local operations to each subsystem A and B. Local operations means a unitary operator that acts only on one of the subsystems, or a measurement that measures an observable on one of the subsystems. Most often, one also allows the observers at A and B to exchange classical information in addition to local operations. The combination is referred to as Local Operations and Classical Communication (LOCC).

A pure state for the system can then be Schmidt decomposed as $|\psi\rangle = \sum_i \sqrt{\alpha_i}|i_A\rangle \otimes |i_B\rangle$, which we will write for short $|\psi\rangle = \sum_i \sqrt{\alpha_i}|ii\rangle$. We use the convention that the Schmidt coefficients α_i are ordered, so that $\alpha_1 \geq \alpha_2 \geq \cdots$. Similarly, we write the second state as $|\phi\rangle = \sum_i \sqrt{\beta_i}|i'i'\rangle$. A vector $\beta = (\beta_1, \ldots, \beta_n)$ is said to majorize another vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ if

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \tag{1}$$

for all k. This is written as $\alpha \prec \beta$.

If we write the fact that $|\psi\rangle$ can be transformed to $|\phi\rangle$ using LOCC as $|\psi\rangle \to |\phi\rangle$, the following theorem (M. Nielsen, Phys. Rev. Lett., **83**, 436 (1999)) gives the necessary and sufficient conditions for one state to be converted to another using LOCC:

$$|\psi\rangle \to |\phi\rangle \quad \text{if and only if} \quad \alpha \prec \beta. \tag{2}$$

a) Show that if both A and B are 2-level systems, then either $|\psi\rangle \to |\phi\rangle$ or $|\phi\rangle \to |\psi\rangle$, or both. That is, one of the states can always be converted to the other. This means that the entanglement is in some sense of the same type in all the states.

b) Show that if both A and B are 2-level systems there exist states that can be converted to any other state using LOCC, and find one example of such a state.

c) What local operations should you apply to transform

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)?$$

How much classical information do you need to transfer?

In general, the local operations that are needed include measurement on one side, with the result of the measurement transmitted as classical information to the other side. We know that if we make a standard projective measurement on one of the particles in an entangled pair, we will end in an eigenstate of the corresponding operator, and entanglement disappears. So if we want to reduce entanglement without eliminating it entirely, we need to make a type of measurement that is affecting the state less (and necessarily giving us less precise information at the same time). One way to achieve this is to let the particle interact and get entangled with another particle, and then measuring on this particle. Consider a 2-level system (we call it system 1) in the state

$$|\psi\rangle_1 = \cos\phi|0\rangle + \sin\phi|1\rangle.$$

We want to make a non-projective measurement of the state by entangling it with a second 2-level system (system 2), which initially is in the state

$$|\chi\rangle_2 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

The entangling operation is given by the unitary transformation which in the tensor product basis $|i\rangle_1 \otimes |j\rangle_2$ is given by the matrix

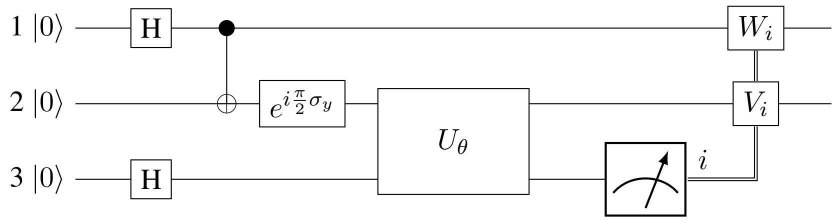
$$U_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{3}$$

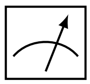
That is, we evolve the system to the final state $U_\theta|\psi\rangle_1 \otimes |\chi\rangle_2$ and measure the second 2-level system.

d) What is the final state of the first particle for each measurement outcome on the second?

e) Give an interpretation of your answer to the previous question. Explain in words what happens physically (Hint: What interaction between the two particles would generate the given unitary transformation U_θ?).

f) We have the following quantum circuit.



Here U_θ is the unitary transformation given in Eq. (3) with the lower line corresponding to the first qubit.  indicates that the qubit is measured in the $\{|0\rangle, |1\rangle\}$-basis, and the rightmost part of the circuit means that the operations V_i and W_i are dependent on the outcome i of the measurement. We choose the operations $V_0 = W_0 = \mathbb{1}$. What operations must V_1 and W_1 be, so that the final state of the system consisting of qubits 1 and 2 is the same, independent of the measurement outcome? What will the final state be?

g) What are the probabilities for each of the measurement outcomes?

h) Describe with words what the different gates in the circuit do and how we can claim that it is realizing the transformation of one state to another using LOCC. Which state is the initial state of the LOCC transformation?

i) Prove that the entropy of entanglement can never be increased using LOCC. If you find a proof, or helpful fact, in the literature, cite your sources.

The following are states of two 3-level systems

$$|\psi\rangle = \sqrt{\frac{1}{2}}|11\rangle + \sqrt{\frac{2}{5}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle$$
$$|\phi\rangle = \sqrt{\frac{3}{5}}|11\rangle + \sqrt{\frac{1}{5}}|22\rangle + \sqrt{\frac{1}{5}}|33\rangle$$

j) Show that neither $|\psi\rangle \to |\phi\rangle$ nor $|\phi\rangle \to |\psi\rangle$. This means that the entanglement in the two states is qualitatively different (in the sense of non-conversion using LOCC).

k) The suggested classification of entanglement based om LOCC is not ideal, as it suffers from at least one serious drawback. Search in the literature and find criticism of this classification. Cite you sources.

One consequence of the theorem (2) is the so-called entanglement catalysis. Local transformations on a composite quantum system can be enhanced in the presence of certain entangled states. These extra states act much like catalysts in a chemical reaction: they allow otherwise impossible local transformations to be realized, without being consumed in any way.

The following are states of two 4-level systems

$$|\psi_1\rangle = \sqrt{0.4}|11\rangle + \sqrt{0.4}|22\rangle + \sqrt{0.1}|33\rangle + \sqrt{0.1}|44\rangle$$
$$|\psi_2\rangle = \sqrt{0.5}|11\rangle + \sqrt{0.25}|22\rangle + \sqrt{0.25}|33\rangle$$

l) Show that neither $|\psi_1\rangle \rightarrow |\psi_2\rangle$ nor $|\psi_2\rangle \rightarrow |\psi_1\rangle$.

m) We assume now that the two parties, in addition to the above 4-level systems also share a pair of entangled 2-level systems in the state

$$|\phi\rangle = \sqrt{0,6}|55\rangle + \sqrt{0.4}|66\rangle$$

Show that

$$|\psi_1\rangle|\phi\rangle \rightarrow |\psi_2\rangle|\phi\rangle$$

This means that the presence of the state $|\phi\rangle$ enables the transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$ without being changed in the process.