FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2020. Solution

Problem 1: Bloch sphere for three-level system

a) Hermitian matrices have real diagonal elements and the lower triangular elements are determined by the upper triangular ones, and are in general complex. This means that we need n^2 real parameters to specify a Hermitian $n \times n$ matrix. The traceless condition reduces this by one, so that the number of λ_i-matrices is $n^2 - 1$.

b) We use the fact that for pure states is $\mathrm{Tr}\,\rho^2 = 1$. We have

$$\mathrm{Tr}\,\rho^2 = \frac{1}{n^2}\,\mathrm{Tr}(\mathbb{1} + 2\alpha m_i \lambda_i + \alpha^2 m_i m_j \lambda_i \lambda_j) = \frac{1}{n^2}(n + 2\alpha^2 |\mathbf{m}|) = 1$$

If we set $|\mathbf{m}| = 1$, we get that

$$\alpha = \sqrt{\frac{n(n-1)}{2}}.$$

c) For n-level systems, the general pure state is $|\psi\rangle = \sum_{i=1}^{n} c_i |i\rangle$, which means that we have n complex coefficients, or $2n$ real coefficients. Normalization reduces the number by one, and the global phase by one, so we have that the space of pure states is $2(n-1)$ dimensional.

d) We have shown that with proper choice of α the pure states have $|\mathbf{m}| = 1$, so they are on the surface of the Bloch sphere. The surface of the Bloch sphere in a space of $n^2 - 1$ dimensions is $n^2 - 2$ dimensional. But the space of pure states is $2(n-1)$ dimensional, and for $n > 2$ is $n^2 - 2 > 2(n-1)$, so the pure states do not cover the whole surface. This means that there are many points on the surface of the Bloch sphere that does not represent any physical quantum state.

e)

$$\rho = \frac{1}{3}\left[\mathbb{1} + \sqrt{3}(m_1\lambda_1 + m_8\lambda_8)\right] = \begin{pmatrix} \frac{1}{3}(1+m_8) & \frac{1}{\sqrt{3}}m_1 & 0 \\ \frac{1}{\sqrt{3}}m_1 & \frac{1}{3}(1+m_8) & 0 \\ 0 & 0 & \frac{1}{3}(1-2m_8) \end{pmatrix}.$$

The eigenvalues are found from

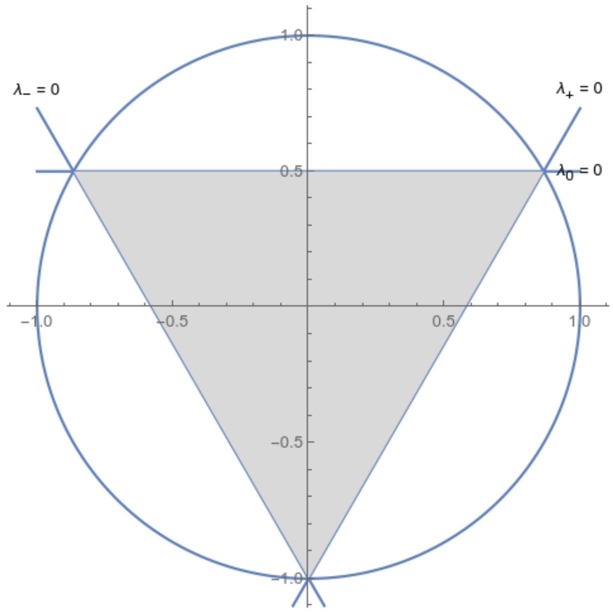
$$\begin{vmatrix} \frac{1}{3}(1+m_8) - \lambda & \frac{1}{\sqrt{3}}m_1 & 0 \\ \frac{1}{\sqrt{3}}m_1 & \frac{1}{3}(1+m_8) - \lambda & 0 \\ 0 & 0 & \frac{1}{3}(1-2m_8) - \lambda \end{vmatrix} = \left[\frac{1}{3}(1-2m_8) - \lambda\right]\left\{\left[\frac{1}{3}(1+m_8) - \lambda\right]^2 - \frac{1}{3}m_1^2\right\} = 0.$$

This gives three possible solutions

$$\lambda_0 = \frac{1}{3}(1 - 2m_8)$$

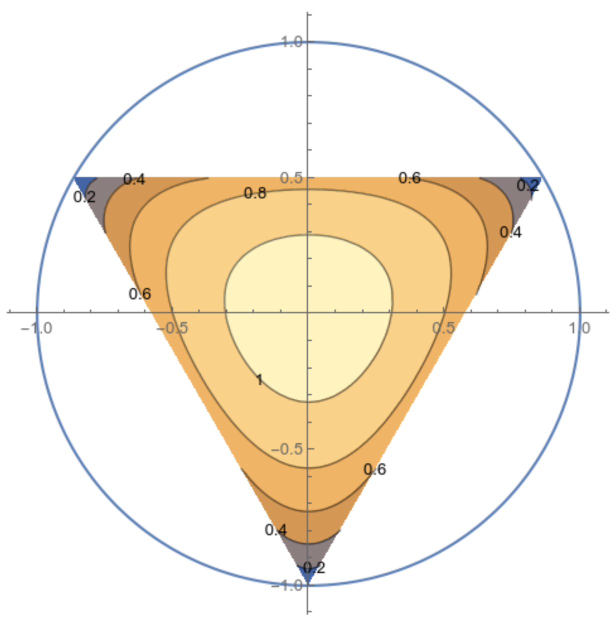
$$\lambda_\pm = \frac{1}{3}(1 + m_8) \pm \frac{1}{\sqrt{3}}m_1$$

f)



g) The entropy is given by

$$S = -\lambda_0 \log \lambda_0 - \lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$$



We can see from the plot that the entropy depends on the direction as well as the length of the Bloch vector.

Problem 2: Entanglement transformations using local operations and classical communication

a) If both A and B are 2-level systems, the vectors α and β have both only two elements. We also know that for the state to be normalized we have $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$. This means that there is only one nontrivial inequality to be considered in the majorization condition [Eq. (1) of the problem set], namely the one for $k = 1$. If $\alpha_1 \le \beta_1$ we have that $\alpha \prec \beta$ and then $|\psi\rangle \to |\phi\rangle$. If $\beta_1 \le \alpha_1$ we have that $\beta \prec \alpha$ and or $|\phi\rangle \to |\psi\rangle$. If $\alpha_1 = \beta_1$ both transformations are possible.

b) The state that is majorized by all other states must have the smallest possible α_1. Since the Schmidt coefficients are assumed to be in decreasing order, this means that $\alpha_1 = \frac{1}{2}$. Any state with this property is majorized by all other states, and consequently can be converted to all other states by LOCC. As an example, we can use one of our familiar Bell states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

c) We apply the unitary transformation σ_z to system A. It has the action

$$\sigma_z|0\rangle = |0\rangle, \qquad \sigma_z|1\rangle = -|1\rangle$$

on the basis states. This gives exactly the specified transformation on the given total state for A and B. No measurements are required, and no classical information has to be transferred.

d) From the given matrix for U_θ we can read the action of the operator on the basis states

$$\begin{aligned}
U_\theta|00\rangle &= \cos\theta|00\rangle - \sin\theta|10\rangle \\
U_\theta|01\rangle &= \cos\theta|01\rangle + \sin\theta|10\rangle \\
U_\theta|10\rangle &= \sin\theta|00\rangle + \cos\theta|10\rangle \\
U_\theta|11\rangle &= -\sin\theta|01\rangle + \cos\theta|11\rangle
\end{aligned}$$

Then we get

$$\begin{aligned}
U_\theta|\psi\rangle_1 \otimes |\chi\rangle_2 = &\frac{1}{\sqrt{2}}\left[(\cos\phi\cos\theta + \sin\phi\sin\theta)|00\rangle + (\cos\phi\cos\theta - \sin\phi\sin\theta)|01\rangle\right. \\
&\left.+(-\cos\phi\sin\theta + \sin\phi\cos\theta)|10\rangle + (\cos\phi\sin\theta + \sin\phi\cos\theta)|11\rangle\right] \\
= &\frac{1}{\sqrt{2}}\left[\cos(\phi-\theta)|00\rangle + \cos(\phi+\theta)|01\rangle + \sin(\phi-\theta)|10\rangle + \sin(\phi+\theta)|11\rangle\right]
\end{aligned}$$

If we measure the second particle, the state of the first particle would be (if we normalize the states)

$$\begin{aligned}
&\text{Measurement outcome 0:} \quad |\psi\rangle_1 = \cos(\phi-\theta)|0\rangle + \sin(\phi-\theta)|1\rangle \\
&\text{Measurement outcome 1:} \quad |\psi\rangle_1 = \cos(\phi+\theta)|0\rangle + \sin(\phi+\theta)|1\rangle
\end{aligned}$$

e) An interaction Hamiltonian of the form $H = -\hbar\omega\sigma_y \otimes \sigma_z$ gives the time evolution $e^{-\frac{i}{\hbar}Ht} = U_{\omega t}$. This mans that the Bloch vector of the first particle will rotate around the y-axis with a direction depedent on the state of the second particle. Since the second particle is in a superposition of the two states, both rotations take place at the same time. When measuring the state of the second particle, the wavefunction collapses, and the corresponding rotation is the only one that is realized.

f) We write the qubits in the order from top to bottom (as indicated by the numbers on the left). Note that we have to be careful when applying the U_θ as it is stated that the lower line should correspond to the first qubit, which is opposite to what we write here. We know that

$$e^{i\frac{\pi}{2}\sigma_y} = \cos\frac{\pi}{2}\mathbb{1} + i\sin\frac{\pi}{2}\sigma_y = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

That is, it exchanges the $|0\rangle$ and $|1\rangle$ states with a change of sign in one case. We then get

$$\begin{aligned}
|000\rangle &\overset{H_1\otimes H_3}{\rightarrow} \frac{1}{2}(|0\rangle + |1\rangle)|0\rangle(|0\rangle + |1\rangle) \\
&\overset{CNOT}{\rightarrow} \frac{1}{2}(|00\rangle + |11\rangle)(|0\rangle + |1\rangle) \\
&\overset{e^{i\frac{\pi}{2}\sigma_y}}{\rightarrow} \frac{1}{2}(|01\rangle - |10\rangle)(|0\rangle + |1\rangle) \\
&\overset{U_\theta}{\rightarrow} \frac{1}{2}\left[|0\rangle\left(\cos\theta|10\rangle + \sin\theta|11\rangle - \sin\theta|10\rangle + \cos\theta|11\rangle\right) - |1\rangle\left(\cos\theta|00\rangle - \sin\theta|01\rangle + \sin\theta|00\rangle + \cos\theta|01\rangle\right)\right] \\
=&\frac{1}{2}\left[(\cos\theta - \sin\theta)|01\rangle - (\cos\theta + \sin\theta)|10\rangle\right]|0\rangle + \frac{1}{2}\left[(\cos\theta + \sin\theta)|01\rangle - (\cos\theta - \sin\theta)|10\rangle\right]|1\rangle
\end{aligned}$$

We now measure the third qubit and get the state of the first two qubits

$$\begin{aligned}
&\text{Measurement outcome 0:} \frac{1}{\sqrt{2}}\left[(\cos\theta - \sin\theta)|01\rangle - (\cos\theta + \sin\theta)|10\rangle\right] \\
&\text{Measurement outcome 1:} \frac{1}{\sqrt{2}}\left[(\cos\theta + \sin\theta)|01\rangle - (\cos\theta - \sin\theta)|10\rangle\right]
\end{aligned}$$

where we have normalized the states. We are told that $V_0 = W_0 = \mathbb{1}$ which means that if the measurement gives 0 we do nothing to any of the first two qubits. To get the same state also if the measurement gives 1, we have to swich the two terms, which we gan achieve by applying $V_1 = W_1 = \sigma_x$.

g) To determine the probabilities of the two outcomes, we need the reduced density matrix of qubit 3. We can rewrite the final state as

$$|\psi\rangle = \frac{1}{2}|01\rangle\left[(\cos\theta - \sin\theta)|0\rangle + (\cos\theta + \sin\theta)|1\rangle\right] - \frac{1}{2}|10\rangle\left[(\cos\theta + \sin\theta)|0\rangle + (\cos\theta - \sin\theta)|1\rangle\right]$$

Then we find that

$$\rho_3 = Tr_{12}\rho = Tr_{12}|\psi\rangle\langle\psi|$$
$$= \frac{1}{4}\left[(\cos\theta - \sin\theta)^2|0\rangle\langle 0| + (\cos^2\theta - \sin^2\theta)(|0\rangle\langle 1| + |1\rangle\langle 0|) + (\cos\theta + \sin\theta)^2|1\rangle\langle 1|\right]$$
$$= \frac{1}{4}\left[(\cos\theta + \sin\theta)^2|0\rangle\langle 0| + (\cos^2\theta - \sin^2\theta)(|0\rangle\langle 1| + |1\rangle\langle 0|) + (\cos\theta - \sin\theta)^2|1\rangle\langle 1|\right] = \frac{1}{2}(|0\rangle\langle 0| + \cos 2\theta(|0\rangle$$

The probabilities for the outcames are

$$P_0 = \mathrm{Tr}(\rho_3|0\rangle\langle 0|) = \frac{1}{2}$$
$$P_1 = \mathrm{Tr}(\rho_3|1\rangle\langle 1|) = \frac{1}{2}.$$

h) The Hadamard gate on the first qubit prepares a superposition of the basis states. The CNOT entangles this with the second qubit. The $e^{i\frac{\pi}{2}\sigma_y}$ flips the second qubit. Together, these three gates prepares the initial state $\frac{1}{\sqrt{2}}(ket01 - |10\rangle)$ that is to be transformed. CNOT is the only gate that is nonlocal in qubits 1 and 2, and they would have to be close enough to intract at that point. Later they are separated, so that qubit 1 is with observer A and qubit 2 with observer B. To execute the transformation, we will measure qubit 2, but only non-projectively. This we do by entangling it with qubit 3 (which we consider to be close to qubit 2, with observer B) in the U_θ-gate and then measuring qubit 3. The outcome of this measurement is used to determine the action V_i on qubit 2 and is sent via classical communication to A to inform about which local unitary W_i should be applied.

i) A proof can be found in M. Nielsen and G. Vidal, Quantum Information and Computation, **1**, 76 (2001). All proofs that I have seen use, like that one, some more general theorem that requires some non-trivial mathematical tools. I have never seen a simple direct proof, but it probably can be found in the literature. The following argument is direct and should make it clear that the entropy can never increase using LOCC.

First, we know that any LOCC process leads to a state with a vector β of squared Schmidt coefficients that majorizes the vector α corresponding to the original state. We also know that the entanglement entropy is given in terms of the vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ by

$$S(\alpha) = -\sum_i \alpha_i \ln \alpha_i.$$

We need therefore to prove that if $\alpha \prec \beta$ then $S(\alpha) \geq S(\beta)$. If we consider the function $-x\ln x$ it has a derivative that is monotonously decreasing. This means that if we increase one of the α_i while decreasing a smaller α by the same amount (remember that $\sum_i \alpha_i = 1$, keeping the rest fixed, the entropy decreases. we need a way to change from α to β so that we always increase a larger α_i and decrease a smaller. Start by incresing α_1 and decreasing α_n until one of the partial sums $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i$. If $k = 1$ or $k = n-1$, we know that $\alpha_1 = \beta_1$ or $\alpha_n = \beta_n$, and we repeat the procedure for the remaining α_i. If the partial sums agree at some intermediate k, we split the vectors at that point, and repeat the procedure for each part independently. We can continue this until $\alpha_i = \beta_i$ for all i.

The following are states of two 3-level systems

$$
\begin{aligned}
|\psi\rangle &= \sqrt{\frac{1}{2}}|11\rangle + \sqrt{\frac{2}{5}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle \\
|\phi\rangle &= \sqrt{\frac{3}{5}}|11\rangle + \sqrt{\frac{1}{5}}|22\rangle + \sqrt{\frac{1}{5}}|33\rangle
\end{aligned}
$$

j) Both the states are already in Schmidt decomposed form, so we read directly that

$$
\begin{aligned}
\alpha_1 &= \frac{1}{2}, \qquad \alpha_2 = \frac{2}{5}, \qquad \alpha_3 = \frac{1}{10} \\
\beta_1 &= \frac{3}{5}, \qquad \beta_2 = \frac{1}{5}, \qquad \beta_3 = \frac{1}{5}
\end{aligned}
$$

From this we find the sums

n	1	2	3
$\sum_{i=1}^{n} \alpha_i$	0.5	0.9	1
$\sum_{i=1}^{n} \beta_i$	0.6	0.8	1

From this we see that we do not have $\alpha \prec \beta$ or $\beta \prec \alpha$ and therefore neither $|\psi\rangle \rightarrow |\phi\rangle$ nor $|\phi\rangle \rightarrow |\psi\rangle$.

k) One problem with classifying different types of entanglement by whether they are convertible using LOCC or not is the fact that states that are very close to each other may be classified as having completely different type of entanglement. One example is given in Martin B. Plenio and S. Virmani, An introduction to entanglement measures, Quant.Inf.Comput. **7**, 1 (2007). The initial state $(|00\rangle + |11\rangle)/sqrt2$ can be transformed by LOCC to $0.8|00\rangle + 0.6|11\rangle$ but not to $(0.8|00\rangle + 0.6|11\rangle + \epsilon|22\rangle)/\sqrt{1+\epsilon^2}$ even if the two final states are arbitrary close for small ϵ. Classification of states according to LOCC transformation does not capture the fact that these states are close. Different modifications have been proposed, where one studies the number of states of one type are needed to get one state of another type, or allows the process to succeed only with a certain probability, see the paper cited above or R. Horodecki et al., Rev. Mod. Phys. **81**, 865 (2009).

l) Since the states already are in Schmidt form, we read directly the vectors α (corresponding to $|\psi_1\rangle$) and β (corresponding to $|\psi_2\rangle$).

$$
\begin{aligned}
\alpha_1 &= 0.4, \qquad \alpha_2 = 0.4, \qquad \alpha_3 = 0.1, \qquad \alpha_4 = 0.1 \\
\beta_1 &= 0.5, \qquad \beta_2 = 0.25, \qquad \beta_3 = 0.25, \qquad \beta_4 = 0
\end{aligned}
$$

From this we find the sums

n	1	2	3	4
$\sum_{i=1}^{n} \alpha_i$	0.4	0.8	0.9	1
$\sum_{i=1}^{n} \beta_i$	0.5	0.75	1	1

From this we see that we do not have $\alpha \prec \beta$ or $\beta \prec \alpha$ and therefore neither $|\psi_1\rangle \rightarrow |\psi_2\rangle$ nor $|\psi_2\rangle \rightarrow |\psi_1\rangle$.

m) We have in total 4 systems, two at A and two at B. A basis for the states of the two systems at A is

$$|ij\rangle_A = |i\rangle_A \otimes |j\rangle_A$$

where $i = 1 \dots 4$ and $j = 5, 6$. The systems at B has a similar basis, and we can then write

$$\begin{aligned}|\psi_1\rangle|\phi\rangle = &\sqrt{0.24}|15\rangle_A \otimes |15\rangle_B + \sqrt{0.24}|25\rangle_A \otimes |25\rangle_B + \sqrt{0.06}|35\rangle_A \otimes |35\rangle_B + \sqrt{0.06}|45\rangle_A \otimes |45\rangle_B \\ &+ \sqrt{0.16}|16\rangle_A \otimes |16\rangle_B + \sqrt{0.16}|26\rangle_A \otimes |26\rangle_B + \sqrt{0.04}|36\rangle_A \otimes |36\rangle_B + \sqrt{0.04}|46\rangle_A \otimes |46\rangle_B.\end{aligned}$$

This is in Schmidt form, and sorting we get the coefficients

$$\begin{aligned}&\alpha_1 = 0.24, \quad \alpha_2 = 0.24, \quad \alpha_3 = 0.16, \quad \alpha_4 = 0.16, \\ &\alpha_5 = 0.06, \quad \alpha_6 = 0.06, \quad \alpha_7 = 0.04, \quad \alpha_8 = 0.04.\end{aligned}$$

Similarly we have

$$\begin{aligned}|\psi_2\rangle|\phi\rangle = &\sqrt{0.3}|15\rangle_A \otimes |15\rangle_B + \sqrt{0.15}|25\rangle_A \otimes |25\rangle_B + \sqrt{0.15}|35\rangle_A \otimes |35\rangle_B \\ &+ \sqrt{0.2}|16\rangle_A \otimes |16\rangle_B + \sqrt{0.1}|26\rangle_A \otimes |26\rangle_B + \sqrt{0.1}|36\rangle_A \otimes |36\rangle_B\end{aligned}$$

$$\begin{aligned}&\beta_1 = 0.3, \quad \beta_2 = 0.2, \quad \beta_3 = 0.15, \quad \beta_4 = 0.15, \\ &\beta_5 = 0.1, \quad \beta_6 = 0.1, \quad \beta_7 = 0, \quad \beta_8 = 0.\end{aligned}$$

From this we find the sums

n	1	2	3	4	5	6	7	8
$\sum_{i=1}^{n} \alpha_i$	0.24	0.48	0.64	0.8	0.86	0.92	0.96	1
$\sum_{i=1}^{n} \beta_i$	0.3	0.5	0.65	0.8	0.9	1	1	1

We see that $\alpha \prec \beta$ which means that $|\psi_1\rangle|\phi\rangle \rightarrow |\psi_2\rangle|\phi\rangle$.