Problem set 10

10.1 Gaussian integrals

$$I = \int_{-\infty}^{\infty} dx e^{-\lambda x^2}$$
$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dy e^{-\lambda\left(x^2+y^2\right)}$$

In polar coordinates:

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-\lambda r^2} dr = 2\pi \int_0^{\infty} r e^{-\lambda r^2} dr = \pi \int_0^{\infty} dr\, 2r e^{-\lambda r^2}$$

Subsituting $u = -\lambda r^2 \Rightarrow -\frac{1}{\lambda} du = 2r\, dr$, we get:

$$I^2 = -\frac{\pi}{\lambda}\int_0^{-\infty} e^u = \frac{\pi}{\lambda} \iff I = \sqrt{\frac{\pi}{\lambda}} \tag{1}$$

Equivalence due to the fact that $e^u > 0$ for all u.

Computing the next integral:

$$I' \equiv \int_{-\infty}^{\infty} dx e^{-\lambda x^2 + ax + b}$$

Completeing the square in the exponent:

$$\begin{aligned}
-\lambda x^2 + ax + b &= -\lambda\left(x^2 - \frac{a}{\lambda}x - \frac{b}{\lambda}\right) \\
&= -\lambda\left(x^2 - \frac{a}{\lambda}x + \left(\frac{a}{2\lambda}\right)^2 - \left(\frac{a}{2\lambda}\right)^2 - \frac{b}{\lambda}\right) \\
&= -\lambda\left(\left(x - \frac{a}{2\lambda}\right)^2 - \left(\frac{a}{2\lambda}\right)^2 - \frac{b}{\lambda}\right) \\
&= -\lambda\left(x - \frac{a}{2\lambda}\right)^2 + \frac{a^2}{4\lambda} + b
\end{aligned}$$

Then we get:

$$I' = \int_{-\infty}^{\infty} dx e^{-\lambda\left(x-\frac{a}{2\lambda}\right)^2 + \frac{a^2}{4\lambda} + b} = e^{\frac{a^2}{4\lambda}+b}\int_{-\infty}^{\infty} dx e^{-\lambda\left(x-\frac{a}{2\lambda}\right)^2}$$

Substituting:

$$u = x - \frac{a}{2\lambda} \Rightarrow du = dx \Longrightarrow I' = e^{\frac{a^2}{4\lambda}+b}\int_{-\infty}^{\infty} e^{-\lambda u^2}$$

Using the result from (1) gives us

$$I' = \sqrt{\frac{\pi}{\lambda}} e^{\frac{a^2}{4\lambda}+b}$$

10.2 Path integral for free particle

a) Consider the bracketed term in the exponent

$$\begin{aligned}(x_1 - x_i)^2 + (x_2 - x_1)^2 &= 2x_1^2 - 2(x_i + x_2)x_1 + x_i^2 + x_2^2 \\ &= 2\left[x_1^2 - (x_i + x_2)x_1 + \frac{(x_i + x_2)^2}{4}\right] + x_i^2 + x_2^2 - \frac{(x_i + x_2)^2}{2} \\ &= 2\left(x_1 - \frac{x_i + x_2}{2}\right)^2 + \frac{1}{2}(x_2 - x_i)^2.\end{aligned}$$

We change the integration variable to $u = x_1 - \frac{x_i + x_2}{2}$ and get

$$I_1 = N_{\Delta t}^2 \int du e^{\frac{im}{2\hbar\Delta t}2u^2} e^{\frac{im}{2\hbar\Delta t}\frac{1}{2}(x_2 - x_i)^2} = \sqrt{\frac{m}{2\pi i\hbar \cdot 2\Delta t}} e^{\frac{im}{2\hbar\cdot 2\Delta t}(x_2 - x_i)^2},$$

where we use Eq (1) and $N_{\Delta t} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}$.

b) The integral over x_2 is similar, in the exponent we will have

$$\frac{1}{2}(x_2 - x_i)^2 + (x_3 - x_2)^2 = \frac{3}{2}\left(x_2 - \frac{x_i + 2x_3}{3}\right)^2 + \frac{1}{3}(x_3 - x_i)^2.$$

We change the integration variable to $u = x_2 - \frac{x_i + 2x_3}{3}$ and get

$$I_2 = \sqrt{\frac{m}{2\pi i\hbar \cdot 3\Delta t}} e^{\frac{im}{2\hbar\cdot 3\Delta t}(x_3 - x_i)^2}.$$

c) We guess that the general form is

$$I_{k-1} = \sqrt{\frac{m}{2\pi i\hbar \cdot k\Delta t}} e^{\frac{im}{2\hbar\cdot k\Delta t}(x_k - x_i)^2}. \tag{2}$$

Then we get

$$I_k = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\sqrt{\frac{m}{2\pi i\hbar \cdot k\Delta t}} \int dx_k e^{\frac{im}{2\hbar\Delta t}\left[\frac{1}{k}(x_k - x_i)^2 + (x_{k+1} - x_k)^2\right]}. \tag{3}$$

For the exponent we find

$$\frac{1}{k}(x_k - x_i)^2 + (x_{k+1} - x_k)^2 = \frac{k+1}{k}\left(x_k - \frac{k}{k+1}(\frac{1}{k}x_i + x_{k+1})\right)^2 + \frac{1}{k+1}(x_{k+1} - x_i)^2 \tag{4}$$

$$I_k = \sqrt{\frac{m}{2\pi i\hbar \cdot (k+1)\Delta t}} e^{\frac{im}{2\hbar\cdot (k+1)\Delta t}(x_{k+1} - x_i)^2}. \tag{5}$$

which is of the same form as (2), and therefore inductively we get

$$I_{n-1} = \int \mathcal{D}x(t) e^{\frac{i}{\hbar}S} = \sqrt{\frac{m}{2\pi i\hbar T}} e^{\frac{im}{2\hbar T}(x_f - x_i)^2}. \tag{6}$$

This is the same as Eq. (1.109) in the lecture notes.

10.3 Path integral for harmonic oscillator

a)

$$x(t) = x_{cl}(t) + \sum_{n=1}^{\infty} c_n \sin\left(n\pi \frac{t - t_i}{T}\right) \tag{7}$$

$$S[x(t)] = \int_{t_i}^{t_f} dt \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right]. \tag{8}$$

The first term is given by (1.106) in the lecture notes, and we have to consider the second term

$$\int_{t_i}^{t_f} dt x^2 = \int_{t_i}^{t_f} dt \left[x_{cl}^2 + \sum_{n,n'} c_n c_{n'} \sin\left(n\pi\frac{t - t_i}{T}\right) \sin\left(n'\pi\frac{t - t_i}{T}\right)\right] = \int_{t_i}^{t_f} dt x_{cl}^2 + \frac{T}{2}\sum_n c_n^2. \tag{9}$$

From this we see that the action can be written

$$S[x(t)] = S[x_{cl}(t)] + \frac{mT}{4}\sum_n \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right] c_n^2.$$

b)

$$\begin{aligned}
\mathcal{G}(x_f t_f, x_i t_i) &= \int \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]} \\
&= N' e^{\frac{i}{\hbar}S[x_{cl}(t)]} \prod_n \int dc_n e^{\frac{mT}{4}\left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right]c_n^2} \\
&= N e^{\frac{i}{\hbar}S[x_{cl}(t)]} \prod_n \left[1 - \left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}
\end{aligned}$$

where we have used that each integral is of the form

$$\int dc_n e^{\frac{mT}{4}\left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right]c_n^2} = \sqrt{\frac{4i\pi\hbar}{mT\left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right]}}$$

and we have collected all the constants from each of these integrals together with N' into the normalization constant N. Using

$$\prod_n \left(1 - \frac{a^2}{n^2}\right) = \frac{\sin a\pi}{a\pi}$$

we get

$$\mathcal{G}(x_f t_f, x_i t_i) = N e^{\frac{i}{\hbar} S[x_{cl}(t)]} \left[\frac{\sin \omega T}{\omega T}\right]^{-1/2}.$$

In the limit $\omega \to 0$ we have $\frac{\sin \omega T}{\omega T} \to 1$ and to recover Eq. (6) we must have

$$N = \sqrt{\frac{m}{2\pi i\hbar T}}$$

so that

$$\mathcal{G}(x_f t_f, x_i t_i) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} e^{\frac{i}{\hbar} S[x_{cl}(t)]}. \tag{10}$$

c) The calculations are simplified if we note the following

$$\int_{t_i}^{t_f} dt \dot{x}_{cl}^2 = [x_{cl}\dot{x}_{cl}]_{t_i}^{t_f} - \int_{t_i}^{t_f} dt x_{cl}\ddot{x}_{cl} = [x_{cl}\dot{x}_{cl}]_{t_i}^{t_f} + \omega^2 \int_{t_i}^{t_f} dt x_{cl}^2 \tag{11}$$

where we used the equation of motion $\ddot{x}_{cl} = -\omega^2 x_{cl}$. The last term in this expression cancels the last term in the action and we find that

$$S[x_{cl}(t)] = \frac{1}{2} m \left[x_{cl}\dot{x}_{cl}\right]_{t_i}^{t_f} \tag{12}$$

The equation of motion has the general solution

$$x(t) = A\cos\omega t + B\sin\omega t \tag{13}$$

The boundary conditions are

$$\begin{aligned} x(t_i) &= A\cos\omega t_i + B\sin\omega t_i = x_i \\ x(t_f) &= A\cos\omega t_f + B\sin\omega t_f = x_f \end{aligned}$$

from which we find

$$A = \frac{x_i \sin\omega t_f - x_f \sin\omega t_i}{\sin\omega T} \qquad B = \frac{x_f\cos\omega t_i - x_i\cos\omega t_f}{\sin\omega T} \tag{14}$$

The classical path is

$$x_{cl} = \frac{x_i \sin\omega(t_f - t) + x_f \sin\omega(t - t_i)}{\sin\omega T} \tag{15}$$

Using (12) we now get

$$S[x_{cl}(t)] = \frac{m\omega}{2\sin\omega T}\left[(x_f^2 + x_i^2)\cos\omega T - 2x_f x_i\right]$$

d) We have

$$\frac{\partial^2 S_{cl}}{\partial x_f \partial x_i} = \frac{m\omega}{\sin \omega T}$$

Inserting into in Eqs. (1.119) and (1.116) of the lecture notes we find

$$\mathcal{G}(x_f t_f, x_i t_i) = \sqrt{\frac{1}{2\pi i\hbar}\left|\frac{\partial^2 S_{cl}}{\partial x_f \partial x_i}\right|} e^{\frac{i}{\hbar}S[x_{cl}(t)]} = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} e^{\frac{i}{\hbar}S[x_{cl}(t)]}.$$

which agrees with Eq (10)

10.4 The Aharonov-Bohm effect

a) We calculate the magnetic field, which will only have a component in the z-direction

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0 \tag{16}$$

if $r > 0$. But the integral of A around a circle of radius r is

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 2\pi k = \Phi \tag{17}$$

which shows that $k = \Phi/2\pi$ and that there has to be an infinite field at the origin to give this flux.

b) In the semiclassical approximation the propagator is given as:

$$p(y) = \lambda \left|G(\mathbf{r}_P, t; \mathbf{r}_S, 0)\right|^2 = \lambda \left| N \sum_{n=1}^{2} e^{iS_n/\hbar} \right|^2 = \lambda \left| N \left(e^{iS_1/\hbar} + e^{iS_2/\hbar} \right) \right|^2$$

Here S_1 and S_2 are the action on classical paths. The action is given by $S\left[\mathbf{r}(\mathbf{t})\right] = \int_0^t \mathcal{L} dt$, and we introduce the difference $\Delta S = S_2 - S_1$, and calculate:

$$\begin{aligned}
p(y) &= \lambda \left|N\right|^2 \left| \left(e^{iS_1/\hbar} + e^{i(\Delta S + S_1)/\hbar} \right) \right|^2 \\
&= \lambda \left| N e^{iS_1/\hbar} \right|^2 \left| \left(1 + e^{i\Delta S/\hbar} \right) \right|^2 \\
&= \lambda \left|N\right|^2 \underbrace{\left| e^{iS_1/\hbar} \right|^2}_{=1} \left(1 + e^{i\Delta S/\hbar} \right) \left(1 + e^{-i\Delta S/\hbar} \right) \\
&= \lambda \left| N e^{iS_1/\hbar} \right|^2 \left(1 + e^{-i\Delta S/\hbar} + e^{i\Delta S/\hbar} + \underbrace{e^{i\Delta S/\hbar} e^{-i\Delta S/\hbar}}_{=1} \right) \\
&= \lambda \left|N\right|^2 \left(2 + e^{-i\Delta S/\hbar} + e^{i\Delta S/\hbar} \right)
\end{aligned}$$

Using the identity $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ gives us:

$$p(y) = 2\lambda\,|N|^2\left(1 + \cos\frac{\Delta S}{\hbar}\right) \tag{18}$$

c) We define the length along path 1 as L_1 and the length along path 2 as L_2, and assume the electrons travel with constant velocities velocities v_1 and v_2 along the respective paths, and that each path has the same length in time. Given the lagrangian $\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\mathbf{A}(\mathbf{r})\cdot\dot{\mathbf{r}}$, the the actions S_1, S_2 along the paths become:

$$\begin{aligned} S_1 &= \int_0^t \frac{1}{2}mv_1^2 dt + e\int_0^t \mathbf{A}(\mathbf{r}_1)\cdot\dot{\mathbf{r}}_1 dt = \frac{1}{2}mv_1^2 t + e\int_S^P \mathbf{A}(\mathbf{r}_1)\cdot d\mathbf{r}_1 = \frac{1}{2}m\frac{L_1^2}{t} + e\int_S^P \mathbf{A}(\mathbf{r}_1)\cdot d\mathbf{r}_1 \\ S_2 &= \int_0^t \frac{1}{2}mv_2^2 dt + e\int_0^t \mathbf{A}(\mathbf{r}_2)\cdot\dot{\mathbf{r}}_2 dt = \frac{1}{2}mv_2^2 t + e\int_S^P \mathbf{A}(\mathbf{r}_2)\cdot d\mathbf{r}_2 = \frac{1}{2}m\frac{L_2^2}{t} + e\int_S^P \mathbf{A}(\mathbf{r}_2)\cdot d\mathbf{r}_2 \end{aligned}$$

Where I used $\int_\ell \mathbf{A}(\mathbf{r})\cdot d\mathbf{r} = \int \mathbf{A}(\mathbf{r})\cdot\frac{d\mathbf{r}}{dt}dt = \int \mathbf{A}(\mathbf{r})\cdot\dot{\mathbf{r}}dt$. Considering the difference ΔS:

$$\begin{aligned} \Delta S &= \frac{1}{2}m\frac{L_2^2}{t} + e\int_S^P \mathbf{A}(\mathbf{r}_2)\cdot d\mathbf{r}_2 - \frac{1}{2}m\frac{L_1^2}{t} - e\int_S^P \mathbf{A}(\mathbf{r}_1)\cdot d\mathbf{r}_1 \\ &= \frac{m}{2t}\left(L_2^2 - L_1^2\right) + e\left(\int_S^P \mathbf{A}(\mathbf{r}_2)\cdot d\mathbf{r}_2 - \int_S^P \mathbf{A}(\mathbf{r}_1)\cdot d\mathbf{r}_1\right) \\ &= \frac{m}{2t}\left(L_2^2 - L_1^2\right) + e\left(\int_S^P \mathbf{A}(\mathbf{r}_2)\cdot d\mathbf{r}_2 + \int_P^S \mathbf{A}(\mathbf{r}_1)\cdot d\mathbf{r}_1\right) \\ &= \frac{m}{t}\bar{L}\Delta L + e\oint_C \mathbf{A}\cdot d\mathbf{r} \end{aligned} \tag{19}$$

We have used that the line integral from S to P over the first path plus the line integral back over another path can be written as a line integral around the curve spanned by the paths, and since the magnetic flux is $\Phi = \oint_C \mathbf{A}\cdot d\mathbf{r}$ this gives:

$$\Delta S = \frac{m}{t}\bar{L}\Delta L + e\Phi \tag{20}$$

Where $\bar{L} = \frac{1}{2}(L_2 + L_1)$ and $\Delta L = (L_2 - L_1)$. This shows it can be written as a function of Φ if we consider the paths fixed.

d) As the vector potential in part a) shows, the vector potential is nonzero everywhere, also outside of the region with magnetic field. So the electron feels directly the presence of the vector potential, which means that it would have to be considered as equally real as the magnetic field. This is curious, since it does not have a specific value at a given point, because this value can be changed by gauge transformations.

e) The Aharonov-Bohm effect does not depend on the choice of gauge becuse the only measurable quantity is the difference of the action integrals, which is only a function of the magnetic flux which is gauge invariant.

f) Inserting (20) into (18) yields:

$$\begin{aligned} p(y) &= 2\lambda\,|N|^2\left(1+\cos\frac{\Delta S}{\hbar}\right) \\ &= 2\lambda\,|N|^2\left(1+\cos\frac{\frac{m}{t}\bar{L}\Delta L+e\Phi}{\hbar}\right) \\ p(y) &= 2\lambda\,|N|^2\left[1+\cos\left(\frac{m\bar{L}\Delta L}{t\hbar}+\frac{e}{\hbar}\Phi\right)\right] \end{aligned} \tag{21}$$

The flux period can be found in two ways:

i) We find the minima/maxima of the cosine, and identify it as half a period:

$$\frac{m\bar{L}\Delta L}{t\hbar}+\frac{e}{\hbar}\Phi_1 = 0 \Rightarrow \Phi_1 = -\frac{m\bar{L}\Delta L}{et} \tag{22}$$

$$\frac{m\bar{L}\Delta L}{t\hbar}+\frac{e}{\hbar}\Phi_2 = \pi \Rightarrow \Phi_2 = \frac{\pi\hbar}{e}-\frac{m\bar{L}\Delta L}{et}$$

The period is then given as

$$T = 2\Delta\Phi = 2\left(\Phi_2-\Phi_1\right) = \frac{2\pi\hbar}{e}$$

ii) We can see the argument in the cosine as some phase $\frac{m\bar{L}\Delta L}{t\hbar}$ (remembering this is constant in our calculations). We then see $e/\hbar$ as a (mathematical) frequency and get:

$$\omega = \frac{2\pi}{T} = \frac{e}{\hbar} \Rightarrow T = \frac{2\pi\hbar}{e}$$

The resulting effect can be seen by looking at the maxima of the propagator in terms of $\bar{L}\Delta L$. From rearranging equation (22), we get

$$\bar{L}\Delta L = -\Phi\frac{et}{m}$$

We see that if $\Phi = 0$, then $\Delta L = 0 \Rightarrow y = 0$ (look at the figure in the exercise sheet). If Φ increses, we see that $\Delta L < 0 \Rightarrow L_1 < L_2$, and if Φ increses in the opposite direction, we get $\Delta L > 0 \Rightarrow L_2 > L_1$. This results in a shift in where the maximas are located, either above or below $y = 0$