# Solutions to problem set 12

### 12.1 Photon emission

a) We know that $\boldsymbol{\epsilon}_{\mathbf{k}a}$ for $a \in \{1,2\}$ are orthonormal vectors in the plane perpendicular to $\mathbf{k}$, and we can write $\mathbf{k} = k\mathbf{e}_k$. Thus, a general (real) vector can be decomposed as follows:

$$\mathbf{a} = (\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})\,\boldsymbol{\epsilon}_{\mathbf{k}1} + (\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})\,\boldsymbol{\epsilon}_{\mathbf{k}2} + (\mathbf{a}\cdot\mathbf{e}_k)\,\mathbf{e}_k$$

Then:

$$\begin{aligned}\sum_a (\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}a})^2 &= (\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})^2 + (\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})^2 \\ &= \underbrace{(\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})^2 + (\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})^2 + (\mathbf{a}\cdot\mathbf{e}_k)^2}_{=\mathbf{a}^2} - (\mathbf{a}\cdot\mathbf{e}_k)^2 \\ &= \mathbf{a}^2 - \left(\mathbf{a}\cdot\frac{\mathbf{k}}{k}\right)^2\end{aligned}$$

b) For the 1D harmonic oscillator, we have the standard ladder operators:

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + i\hat{p}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} - i\hat{p}\right)$$

This gives the momentum operator:

$$\begin{aligned}\hat{p} &= \frac{1}{2i}\left(\sqrt{2m\hbar\omega}\hat{a} - m\omega\hat{x} + m\omega\hat{x} - \sqrt{2m\hbar\omega}\hat{a}^\dagger\right) \\ &= -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right)\end{aligned}$$

These ladder operators act on the $|n\rangle$ states.

$$\begin{aligned}\langle n-1|\hat{p}|n\rangle &= -i\sqrt{\frac{m\hbar\omega}{2}}\langle n-1|\left(\hat{a} - \hat{a}^\dagger\right)|n\rangle \\ &= -i\sqrt{\frac{m\hbar\omega}{2}}\langle n-1|\hat{a}|n\rangle - \langle n-1|\hat{a}^\dagger|n\rangle \\ &= -i\sqrt{\frac{m\hbar\omega}{2}}\langle n-1|\sqrt{n}|n-1\rangle - \underbrace{\langle n-1|\sqrt{n+1}|n+1\rangle}_{=0} \\ &= -i\sqrt{\frac{m\hbar\omega n}{2}}\end{aligned}$$

Where we note that this is in the $z$-direction as noted in the exercise text.

c) From the text:

$$\begin{aligned} p(\theta,\phi) &= \kappa \sum_a \mid \langle n-1, 1_{\mathbf{k}a} | \hat{H}_{emis} | n, 0\rangle \mid^2 \\ &= \kappa \sum_a \mid \langle n-1, 1_{\mathbf{k}a} | -\frac{e}{m} \sum_{\mathbf{k}'a'} \sqrt{\frac{\hbar}{2V\epsilon_0\omega}} \hat{\mathbf{p}} \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'} \hat{a}^\dagger_{\mathbf{k}'a'} | n, 0\rangle \mid^2 \\ &= \kappa \mid \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}} \mid^2 \sum_a \sum_{\mathbf{k}'a'} \mid \langle n-1, 1_{\mathbf{k}a} | \hat{\mathbf{p}} \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'} \hat{a}^\dagger_{\mathbf{k}'a'} | n, 0\rangle \mid^2 \end{aligned}$$

We note that the ladder operator in this case acts on the photon states, while $\hat{\mathbf{p}}$ acts on the particle states. Further:

$$\begin{aligned} \langle n-1, 1_{\mathbf{k}a} | \hat{\mathbf{p}} \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'} \hat{a}^\dagger_{\mathbf{k}'a'} | n, 0\rangle &= \langle n-1| \otimes \langle 1_{\mathbf{k}a}| \left(\hat{\mathbf{p}} \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'} \otimes \mathbb{1}\right) \left(\mathbb{1} \otimes \hat{a}^\dagger_{\mathbf{k}'a'}\right) |n\rangle \otimes |0\rangle \\ &= \left[\left(\langle n-1|\hat{\mathbf{p}} \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'}\right) \otimes \langle 1_{\mathbf{k}a}|\right] \left[|n\rangle \otimes \left(\hat{a}^\dagger_{\mathbf{k}'a'}|0\rangle\right)\right] \\ &= \langle n-1|\hat{\mathbf{p}} \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'}|n\rangle \underbrace{\langle 1_{\mathbf{k}a}|\hat{a}^\dagger_{\mathbf{k}'a'}|0\rangle}_{=\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}} \\ &= \langle n-1|\hat{\mathbf{p}}|n\rangle \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'} \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \end{aligned}$$

Where I marked the quantum numbers in order to have the possibility for the photon state to differ from the one in the emmision hamiltonian. So far, we have:

$$\begin{aligned} p(\theta,\phi) &= \kappa \mid \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}} \mid^2 \sum_a \sum_{\mathbf{k}'a'} \mid \langle n-1|\hat{\mathbf{p}}|n\rangle \cdot \boldsymbol{\epsilon}_{\mathbf{k}'a'} \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \mid^2 \\ &= \kappa \mid \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}} \mid^2 \sum_a \mid \langle n-1|\hat{\mathbf{p}}|n\rangle \cdot \boldsymbol{\epsilon}_{\mathbf{k}a} \mid^2 \end{aligned}$$

Using the result from b) yields:

$$\begin{aligned} p(\theta,\phi) &= \kappa \mid \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}} \mid^2 \sum_a \mid -i\sqrt{\frac{m\hbar\omega n}{2}} \mathbf{e}_z \cdot \boldsymbol{\epsilon}_{\mathbf{k}a} \mid^2 \\ &= \kappa \mid \frac{e}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega}} \sqrt{\frac{m\hbar\omega n}{2}} \mid^2 \sum_a \mid \mathbf{e}_z \cdot \boldsymbol{\epsilon}_{\mathbf{k}a} \mid^2 \end{aligned}$$

Noting that both $\mathbf{e}_z$ and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ are real, we see that it is on the same form as the identity in exercise a).

$$\begin{aligned} p(\theta,\phi) &= \frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(\mathbf{e}_z^2 - \left(\mathbf{e}_z \cdot \frac{\mathbf{k}}{k}\right)^2\right) \\ &= \frac{\kappa e^2\hbar^2 n}{4mV\epsilon_0}\left(1 - \left(\mathbf{e}_z \cdot \frac{\mathbf{k}}{k}\right)^2\right) \end{aligned}$$

The probability is given in terms of angular coordinates, and $\mathbf{k}/k$ has unit length, which allows us to write:

$$\frac{\mathbf{k}}{k} = \cos\phi\sin\theta\mathbf{e}_x + \sin\phi\sin\theta\mathbf{e}_y + \cos\theta\mathbf{e}_z$$

Inserting this yields:

$$p(\theta, \phi) = \frac{\kappa e^2 \hbar^2 n}{4mV\epsilon_0}\left(1 - \cos^2\theta\right)$$

This expression may or may not be normalized, we check:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, p(\theta, \phi) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, \frac{\kappa e^2 \hbar^2 n}{4mV\epsilon_0}\left(1 - \cos^2\theta\right)$$

Using the substitution:

$$\sin\theta d\theta = -\frac{d\cos\theta}{d\theta} d\theta = -d\cos\theta$$

the boundries change to $\cos 0 = 1$ to $\cos\pi = -1$, and the minus interchanges the boundaries:

$$\begin{aligned}
\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, p(\theta, \phi) &= \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, \frac{\kappa e^2 \hbar^2 n}{4mV\epsilon_0}\left(1 - \cos^2\theta\right) \\
&= 2\pi \frac{\kappa e^2 \hbar^2 n}{4mV\epsilon_0} \int_{-1}^{1} d\cos\theta \left(1 - \cos^2\theta\right) \\
&= \frac{\pi\kappa e^2 \hbar^2 n}{2mV\epsilon_0}\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_{\cos\theta=-1}^{\cos\theta=1} \\
&= \frac{\pi\kappa e^2 \hbar^2 n}{2mV\epsilon_0}\left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] \\
&= \frac{\pi\kappa e^2 \hbar^2 n}{2mV\epsilon_0}\left[2 - \frac{2}{3}\right] \\
&= \frac{2}{3}\frac{\pi\kappa e^2 \hbar^2 n}{mV\epsilon_0}
\end{aligned}$$

Then, the normalized probability becomes:

$$\begin{aligned}
p(\theta, \phi) &= \frac{3}{2}\frac{mV\epsilon_0}{\pi\kappa e^2 \hbar^2 n}\frac{\kappa e^2 \hbar^2 n}{4mV\epsilon_0}\left(1 - \cos^2\theta\right) \\
&= \frac{3}{8\pi}\left(1 - \cos^2\theta\right)
\end{aligned}$$

This also seems resonable as the probabilities now only depent on angles, and not on things as the charge, potential and what excitation the charge is in.

## 12.2 Electric dipole transition in hydrogen (Exam 2008)

See solutions to previous exam questions.

## 12.3 Spinflip radiation

a) The probability distribution for the direction of the emitted photon, $p(\theta, \phi)$ is according to the problem given by

$$p(\theta, \phi) = N \sum_a |(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}a}) \cdot \boldsymbol{\sigma}_{BA}|^2$$

where $N$ is a normalization to be determined later. We have

$$\boldsymbol{\sigma}_{BA} = \langle \downarrow | \boldsymbol{\sigma} | \uparrow \rangle = (1, i, 0)$$

We can choose the polarization vectors so that

$$\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}1} = k\boldsymbol{\epsilon}_{\mathbf{k}2} \qquad \text{and} \qquad \mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}2} = -k\boldsymbol{\epsilon}_{\mathbf{k}1}$$

where $k = |\mathbf{k}|$. This means that we get

$$\sum_a |(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}a}) \cdot \boldsymbol{\sigma}_{BA}|^2 = k^2 \sum_a |\boldsymbol{\epsilon}_{\mathbf{k}a} \cdot \boldsymbol{\sigma}_{BA}|^2 = k^2(|\boldsymbol{\sigma}_{BA}|^2 - |\boldsymbol{\sigma}_{BA} \cdot \frac{\mathbf{k}}{k}|^2$$

Using

$$\frac{\mathbf{k}}{k} = (\sin\theta\cos\phi\sin\theta\sin\phi, \cos\theta)$$

we get

$$|\boldsymbol{\sigma}_{BA}|^2 = 2 \qquad \text{and} \qquad \boldsymbol{\sigma}_{BA} \cdot \frac{\mathbf{k}}{k} = \sin\theta e^{i\phi}.$$

The probability distribution for the direction of the emitted photon is then

$$p(\theta, \phi) = Nk^2(1 + \cos^2\theta).$$

To determine the normalization we calculate

$$\int d\phi \int d\theta \sin\theta p(\theta, \phi) = Nk^2 2\pi \int_0^\pi d\theta \sin\theta(1 + \cos^2\theta) = \frac{16\pi}{3} Nk^2 = 1$$

From which we get $N = 3/16\pi k^2$. The answer is then

$$p(\theta, \phi) = \frac{3}{16\pi}(1 + \cos^2\theta).$$

b) We have $\mathbf{k} = (1, 0, 0)$ and we can chosse the polarization vectors so that $\boldsymbol{\epsilon}_{\mathbf{k}1} = (0, \cos\alpha, \sin\alpha)$. Then

$$p(\alpha) = N|(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}1}) \cdot \boldsymbol{\sigma}_{BA}|^2 = N\sin^2\alpha$$

To determine the normalization, we use the condition $\int_0^\pi d\alpha p(\alpha) = 1$. Here we set the upper limit of the integrtion to $\pi$ and not $2\pi$ since $\alpha$ and $\alpha + \pi$ represents the same polarization state. We then get

$$p(\alpha) = \frac{2}{\pi}\sin^2\alpha$$

c)

$$
\begin{aligned}
w_{BA} &= \frac{V}{(2\pi\hbar)^2}\int d^3k \sum_a |\langle B, 1_{\mathbf{k}a}|H_1|A,0\rangle|^2\delta(\omega-\omega_B) \\
&= \frac{e^2\hbar}{32\pi^2 m^2\epsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} k^2 dk \frac{1}{\omega}\delta(\omega-\omega_B)\sum_a |(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}a})\cdot\boldsymbol{\sigma}_{BA}|^2 \\
&= \frac{e^2\hbar}{32\pi^2 m^2\epsilon_0 c^5} 2\pi \int_0^{\pi} d\theta \sin\theta (1+\cos^2\theta) \int_0^{\infty} d\omega \omega^3 \delta(\omega-\omega_B) \\
&= \frac{e^2\hbar\omega_B^3}{6\pi m^2\epsilon_0 c^5}.
\end{aligned}
$$

This gives the lifetime

$$
\tau = \frac{1}{w_{BA}} = \frac{6\pi m^2\epsilon_0 c^5}{e^2\hbar\omega_B^3}.
$$