

Solutions to problem set 1

1.1 Commutators and anti-commutators

We have

$$\begin{aligned} [A, BC] &= ABC - BAC + BAC - BCA = [A, B]C + B[A, C] \\ [A, BC] &= ABC + BAC - BAC - BCA = \{A, B\}C - B\{A, C\} \end{aligned} \quad (1)$$

1.2 Trace and determinant

a) We have

$$A_{mn} = \langle m | \hat{A} | n \rangle \quad A'_{mn} = \langle m' | \hat{A} | n' \rangle \quad (2)$$

and

$$|n\rangle = \sum_n U_{n'n} |n'\rangle \quad U_{n'n} = \langle n | n' \rangle \quad (3)$$

with U a unitary matrix. Then we get

$$A'_{mn} = \sum_{ij} \langle m | i \rangle \langle i | \hat{A} | j \rangle \langle j | n \rangle = (U^{-1} A U)_{mn} \quad (4)$$

From this we find

$$\text{Tr } A' = \text{Tr}(U^{-1} A U) = \text{Tr}(U U^{-1} A) = \text{Tr } A \quad (5)$$

where we use that $\text{Tr } AB = \text{Tr } BA$ and that $U U^{-1} = 1$ (U is unitary).

$$\det A' = \det(U^{-1} A U) = \det U^{-1} \det A \det U = \det A \quad (6)$$

since $\det(AB) = \det A \det B$ and $\det U^{-1} = (\det U)^{-1}$.

b) We write \hat{A} in the basis of eigenstates, where it is diagonal with the eigenvalues a_n , $n = 1, 2, \dots$ on the diagonal. Then the trace is just the sum of the diagonal elements, and the determinant the product.

$$\text{Tr } \hat{A} = \sum_n a_n \quad \det \hat{A} = \prod_n a_n \quad (7)$$

c)

$$\det e^{\hat{A}} = \det \sum_n e^{a_n} |n\rangle \langle n| = \prod_n e^{a_n} = e^{\sum_n a_n} = e^{\text{Tr } \hat{A}} \quad (8)$$

d) We decompose the states in a basis:

$$|\psi\rangle = \sum_n \psi_n |n\rangle \quad |\phi\rangle = \sum_n \phi_n |n\rangle. \quad (9)$$

Then we get

$$\langle\psi|\phi\rangle = \sum_{m,n} \langle n|\psi_n^* \phi_m |m\rangle = \sum_{m,n} \psi_n^* \phi_m \langle n|m\rangle = \sum_n \psi_n^* \phi_n \quad (10)$$

$$\text{Tr}(|\phi\rangle\langle\psi|) = \text{Tr}\left(\sum_{m,n} \psi_n^* \phi_m |m\rangle\langle n|\right) = \sum_n \psi_n^* \phi_n = \langle\psi|\phi\rangle. \quad (11)$$

1.3 Dirac's delta function

a) We start from the definition of the delta function

$$f(x) = \int_{-\infty}^{\infty} dx' \delta(x - x') f(x') \quad (12)$$

and Fourier transform both sides:

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx} \int dx' \delta(x - x') f(x') \\ &= \frac{1}{\sqrt{2\pi}} \int du \int dx' e^{-iku - ikx'} \delta(u) f(x') \\ &= \int du e^{-iku} \delta(u) \tilde{f}(k) \end{aligned} \quad (13)$$

where $u = x - x'$. Thus we must have that

$$\int du e^{-iku} \delta(u) = 1 \quad (14)$$

which means that the Fourier transform of the delta function is a constant

$$\tilde{\delta}(k) = \frac{1}{\sqrt{2\pi}} \quad (15)$$

Using the expression for the inverse Fourier transform we get that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \quad (16)$$

- b) Consider first the case where $g(x)$ has a single zero, $g(x_0) = 0$ and that $g'(x_0) > 0$. Change integration variable to $u = g(x)$ so that $dx = \frac{du}{g'(x)}$:

$$\begin{aligned} \int_{-\infty}^{\infty} dx \delta(g(x)) f(x) &= \int_{-\infty}^{\infty} \frac{du}{g'(x)} \delta(u) f(x) \\ &= \frac{1}{g'(x_0)} f(x_0) = \int_{-\infty}^{\infty} \frac{dx}{g'(x_0)} \delta(x - x_0) f(x) \end{aligned} \quad (17)$$

If instead $g'(x_0) < 0$ we have

$$\int_{-\infty}^{\infty} dx \delta(g(x)) f(x) = \int_{\infty}^{-\infty} \frac{du}{g'(x)} \delta(u) f(x) = - \int_{-\infty}^{\infty} \frac{du}{g'(x)} \delta(u) f(x) \quad (18)$$

Both cases can then be written in the form

$$\delta(g(x)) = \frac{1}{|g'(x_0)|} \delta(x - x_0) \quad (19)$$

If the function $g(x)$ has several zeros, at the points $x = x_i$ we get one contribution from each zero, which gives the general formula

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i) \quad (20)$$

1.4 Position and momentum eigenstates

Consider the momentum eigenstate $|p\rangle$. In the coordinate representation it is given by the wavefunction $\psi_p(x) = \langle x|p\rangle$ which is exactly the scalar product we have to find. The eigenvalue equation $\hat{p}|p\rangle = p|p\rangle$ is in the coordinate representation

$$\langle x|\hat{p}|p\rangle = p\langle x|p\rangle = p\psi_p(x) \quad (21)$$

The left hand side of this equation is the position basis representation of state that results from the action of the momentum operator on the state $|p\rangle$. This we know from introductory quantum mechanics to be (remember that $\psi_p(x) = \langle x|p\rangle$ is the position space representation of the momentum eigenstate)

$$\langle x|\hat{p}|p\rangle = -i\hbar \frac{d\psi_p(x)}{dx} \quad (22)$$

In fact, this equation is the proper meaning of the prescription $\hat{p} \rightarrow -i\hbar \frac{d}{dx}$ that is used in the position representation. Thus we get the differential equation

$$-i\hbar \frac{d\psi_p(x)}{dx} = p\psi_p(x) \quad (23)$$

with the solution

$$p(x) = Ae^{\frac{i}{\hbar}xp} \quad (24)$$

where A is an integration constant which we have to determine from the normalization condition

$$\langle p|p' \rangle = \int dx \langle p|x \rangle \langle x|p' \rangle = \int dx |A|^2 e^{\frac{i}{\hbar}x(p-p')} = 2\pi\hbar |A|^2 \delta(p-p'). \quad (25)$$

So we have to choose $A = 1/\sqrt{2\pi\hbar}$ and get

$$\langle x|p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}xp} \quad (26)$$

1.5 Some operator expansions

We're given the following generally noncommuting operators:

$$\hat{F}(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}, \quad \hat{G}(\lambda) = e^{\lambda\hat{A}}e^{\lambda\hat{B}}$$

a)

$$\begin{aligned} \frac{d\hat{F}}{d\lambda} &= \frac{de^{\lambda\hat{A}}}{d\lambda}\hat{B}e^{-\lambda\hat{A}} + e^{\lambda\hat{A}}\underbrace{\frac{d\hat{B}}{d\lambda}}_{=0}e^{-\lambda\hat{A}} + e^{\lambda\hat{A}}\hat{B}\frac{de^{-\lambda\hat{A}}}{d\lambda} \\ &= \hat{A}e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} - e^{\lambda\hat{A}}\hat{B}\hat{A}e^{-\lambda\hat{A}} \end{aligned}$$

Remembering that $e^{\lambda\hat{A}}$ can be written as the Taylor expansion evaluated for small λ , we know that $[e^{k\hat{A}}, \hat{A}] = 0$ since every operator commutes with itself.

$$\frac{d\hat{F}}{d\lambda} = \hat{A}e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} - e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}\hat{A} = \hat{A}\hat{F} - \hat{F}\hat{A} = [\hat{A}, \hat{F}] \quad (27)$$

Then expanding \hat{F} for small λ , we get:

$$\begin{aligned} \hat{F}(\lambda) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \hat{F}^{(n)}(0) \\ &= \hat{F}(0) + \lambda\hat{F}'(0) + \frac{\lambda^2}{2}\hat{F}''(0) + \dots \\ &= \hat{F}(0) + \lambda[\hat{A}, \hat{F}(0)] + \frac{\lambda^2}{2}[\hat{A}, \hat{F}'(0)] + \dots \\ &= \hat{F}(0) + \lambda[\hat{A}, \hat{F}(0)] + \frac{\lambda^2}{2}[\hat{A}, [\hat{A}, \hat{F}(0)]] + \dots \\ &= \hat{B} + \lambda[\hat{A}, \hat{B}] + \frac{\lambda^2}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \dots \quad \because \hat{F}(0) = \hat{B} \end{aligned} \quad (28)$$

b)

$$\begin{aligned}
 \frac{d\hat{G}}{d\lambda} &= \frac{de^{\lambda\hat{A}}}{d\lambda} e^{\lambda\hat{B}} + e^{\lambda\hat{A}} \frac{de^{\lambda\hat{B}}}{d\lambda} \\
 &= \hat{A}e^{\lambda\hat{A}} e^{\lambda\hat{B}} + e^{\lambda\hat{A}} \hat{B}e^{\lambda\hat{B}} \\
 &= \hat{A}e^{\lambda\hat{A}} e^{\lambda\hat{B}} + e^{\lambda\hat{A}} \hat{B}e^{-\lambda\hat{A}} e^{\lambda\hat{A}} e^{\lambda\hat{B}} \\
 &= \hat{A}\hat{G} + \hat{F}\hat{G} \\
 &= (\hat{A} + \hat{F})\hat{G}
 \end{aligned} \tag{29}$$

Then, we need to show the relation

$$\hat{G}(\lambda) = e^{\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}] + \dots} \tag{30}$$

We're told to compute the exponential to second order in λ . If we then treat $\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[A, B]$ as a single variable, we can easily see that we can expand in the following way for small λ :

$$e^{\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}]} = \sum_{n=0}^{\infty} \frac{\left(\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[A, B]\right)^n}{n!} = 1 + \lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}] + \frac{1}{2}\left(\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}]\right)^2 + \dots$$

Neglecting terms that are of $\mathcal{O}(\lambda^3)$, we get:

$$\begin{aligned}
 e^{\lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}]} &= 1 + \lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}] + \frac{1}{2}(\lambda\hat{A} + \lambda\hat{B})^2 + \dots \\
 &= 1 + \lambda\hat{A} + \lambda\hat{B} + \frac{\lambda^2}{2}\left([\hat{A}, \hat{B}] + (\hat{A} + \hat{B})^2\right) + \dots
 \end{aligned} \tag{31}$$

If we now go on to expand the left hand side of (30) to second order in λ for small λ , we get:

$$\hat{G}(\lambda) = \hat{G}(0) + \lambda\hat{G}'(0) + \frac{\lambda^2}{2}\hat{G}''(0) + \dots$$

Calculating the second order term:

$$\frac{d^2\hat{G}}{d\lambda^2} = \frac{d}{d\lambda} \left(\hat{A} + \hat{F}\right) \hat{G} = \underbrace{\left(\frac{d\hat{A}}{d\lambda} + \frac{d\hat{F}}{d\lambda}\right)}_{=0} \hat{G} + (\hat{A} + \hat{F}) \frac{d\hat{G}}{d\lambda} = [\hat{A}, \hat{F}] \hat{G} + (\hat{A} + \hat{F})^2 \hat{G}$$

Inserting this back yields:

$$\begin{aligned}
 \hat{G}(\lambda) &= \hat{G}(0) + \lambda(\hat{A} + \hat{F}(0))\hat{G}(0) + \frac{\lambda^2}{2}\left([\hat{A}, \hat{F}(0)]\hat{G}(0) + (\hat{A} + \hat{F}(0))^2\hat{G}(0)\right) + \dots \\
 &= 1 + \lambda(\hat{A} + \hat{B}) + \frac{\lambda^2}{2}\left([\hat{A}, \hat{B}] + (\hat{A} + \hat{B})^2\right) + \dots
 \end{aligned} \tag{32}$$

Comparing (31) and (32), we see they are the same. Q.E.D

c) If $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, then from (28), we get $\hat{F}(\lambda) = 1 + \lambda [\hat{A}, \hat{B}]$. Using this and (29), we get

$$\begin{aligned}\frac{d\hat{G}}{d\lambda} &= (\hat{A} + \hat{B} + \lambda [\hat{A}, \hat{B}]) \hat{G} \\ \frac{1}{\hat{G}} \frac{d\hat{G}}{d\lambda} &= (\hat{A} + \hat{B} + \lambda [\hat{A}, \hat{B}])\end{aligned}$$

Integrating over λ yields (assuming $\lambda > 0$):

$$\begin{aligned}\int \frac{1}{\hat{G}} d\hat{G} &= \int (\hat{A} + \hat{B} + \lambda [\hat{A}, \hat{B}]) d\lambda \\ \ln \hat{G} &= \lambda (\hat{A} + \hat{B}) + \frac{\lambda^2}{2} [\hat{A}, \hat{B}] + C \\ \hat{G}(\lambda) &= e^{\lambda(\hat{A}+\hat{B}) + \frac{\lambda^2}{2}[\hat{A},\hat{B}] + C}\end{aligned}$$

Using $\hat{G}(0) = 1$, we get $C = 0$, and:

$$\hat{G}(\lambda) = e^{\lambda(\hat{A}+\hat{B}) + \frac{\lambda^2}{2}[\hat{A},\hat{B}]}$$

Q.E.D

1.6 Spin operators and Pauli matrices

a) We set $\mathbf{n} = (n_1, n_2, n_3)$, then

$$\mathbf{n} \cdot \boldsymbol{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 = \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix} \quad (33)$$

The eigenvalue equation is:

$$\det(\boldsymbol{\sigma}_{\mathbf{n}} - \lambda \mathbb{1}) = 0 \Rightarrow \det \begin{pmatrix} n_3 - \lambda & n_1 - in_2 \\ n_1 + in_2 & -n_3 - \lambda \end{pmatrix} = 0$$

$$\begin{aligned}-(n_3 - \lambda)(n_3 + \lambda) - |n_1 + in_2|^2 &= 0 \\ \Rightarrow -n_3^2 + \lambda^2 - (n_1^2 + n_2^2) &= 0 \\ \Rightarrow \lambda^2 &= n_1^2 + n_2^2 + n_3^2 \\ \Rightarrow \lambda &= \pm |\mathbf{n}| = \pm 1\end{aligned}$$

The corresponding eigenstate equation for $\lambda = 1$ is:

$$\boldsymbol{\sigma}_{\mathbf{n}} \Psi_{\mathbf{n}} = \Psi_{\mathbf{n}}$$

By switching to spherical coordinates, we get $\mathbf{n} = (n_1, n_2, n_3) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$. Inserting this into the matrix (33), the equation gets the form:

$$\begin{pmatrix} \cos \theta & (\cos \phi - i \sin \phi) \sin \theta \\ (\cos \phi + i \sin \phi) \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Which corresponds to these equations:

$$\begin{aligned} \psi_1 \cos \theta + \psi_2 e^{-i\phi} \sin \theta &= \psi_1 \\ \psi_1 e^{i\phi} \sin \theta - \psi_2 \cos \theta &= \psi_2 \end{aligned}$$

We need to solve one of them, I'm choosing the second, where I get the relation:

$$\frac{e^{i\phi} \sin \theta}{1 + \cos \theta} \psi_1 = \psi_2$$

Then:

$$\frac{e^{i\phi} \sin \theta}{1 + \cos \theta} = \frac{e^{i\phi} 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{e^{i\phi} \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}$$

This gives us the relation:

$$e^{i\phi} \sin \frac{\theta}{2} \psi_1 = \psi_2 \cos \frac{\theta}{2}$$

This is true for $\psi_1 = \cos \frac{\theta}{2}$ and $\psi_2 = e^{i\phi} \sin \frac{\theta}{2}$, which defines the eigenvectors for $\lambda = 1$:

$$\Psi_{\mathbf{n}} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}$$

For $\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \mathbf{n}$, we get:

$$\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \left(\Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}} \right)$$

$$\begin{aligned} \Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}} &= \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \\ &= \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2} \right) \begin{pmatrix} e^{i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix} \\ &= e^{i\phi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + e^{-i\phi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ &= \frac{1}{2} \sin \theta \left(e^{i\phi} + e^{-i\phi} \right) \\ &= \cos \phi \sin \theta = n_1 \end{aligned}$$

$$\begin{aligned} \Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}} &= \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2} \right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \\ &= \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2} \right) \begin{pmatrix} -ie^{i\phi} \sin \frac{\theta}{2} \\ i \cos \frac{\theta}{2} \end{pmatrix} \\ &= i \left(-e^{i\phi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + e^{-i\phi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) \\ &= \frac{1}{2} i \sin \theta \left(e^{-i\phi} - e^{i\phi} \right) \\ &= -i^2 \sin \theta \sin \phi = n_2 \end{aligned}$$

$$\begin{aligned}
 \Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}} &= \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \\
 &= \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2} \right) \begin{pmatrix} \cos \frac{\theta}{2} \\ -e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \\
 &= \cos^2 \frac{\theta}{2} - e^{-i\phi} e^{i\phi} \sin^2 \frac{\theta}{2} \\
 &= \cos \theta = n_3
 \end{aligned}$$

Thus:

$$\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \left(\Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}} \right) = (n_1, n_2, n_3) = \mathbf{n}$$

b)

$$e^{-\frac{i}{2}\alpha\sigma_z} \sigma_x e^{\frac{i}{2}\alpha\sigma_z}$$

We have:

$$e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}} = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2} [\hat{A}, [\hat{A}, \hat{B}]] + \dots \quad (34)$$

We choose $\hat{A} = -\frac{i}{2}\sigma_z$, we get our expression on the same form as in (34):

$$\begin{aligned}
 e^{-\frac{i}{2}\alpha\sigma_z} \sigma_x e^{\frac{i}{2}\alpha\sigma_z} &= \sigma_x + \alpha \left[-\frac{i}{2}\sigma_z, \sigma_x \right] + \frac{\alpha^2}{2} \left[-\frac{i}{2}\sigma_z, \left[-\frac{i}{2}\sigma_z, \sigma_x \right] \right] + \frac{\alpha^3}{6} \left[-\frac{i}{2}\sigma_z, \left[-\frac{i}{2}\sigma_z, \left[-\frac{i}{2}\sigma_z, \sigma_x \right] \right] \right] + \dots \\
 &= \sigma_x + \alpha \left(-\frac{i}{2} \right) [\sigma_z, \sigma_x] + \frac{\alpha^2}{2} \left(-\frac{i}{2} \right)^2 [\sigma_z, [\sigma_z, \sigma_x]] + \frac{\alpha^3}{6} \left(-\frac{i}{2} \right)^3 [\sigma_z, [\sigma_z, [\sigma_z, \sigma_x]]] + \dots
 \end{aligned}$$

Then we know that the commutators are such that $[\sigma_x, \sigma_y] = 2i\epsilon_{xyz}\sigma_z$ where $\epsilon_{xyz} = 1$ and any odd number of permutations returns -1 , and any even number of permutations return 1 . Thus: $[\sigma_z, \sigma_x] = 2i\sigma_y$, $[\sigma_z, \sigma_y] = -2i\sigma_x$, and

$$\begin{aligned}
 e^{-\frac{i}{2}\alpha\sigma_z} \sigma_x e^{\frac{i}{2}\alpha\sigma_z} &= \sigma_x + \alpha \left(-\frac{i}{2} \right) 2i\sigma_y + \frac{\alpha^2}{2} \left(-\frac{i}{2} \right)^2 (2i) [\sigma_z, \sigma_y] \\
 &\quad + \frac{\alpha^3}{6} \left(-\frac{i}{2} \right)^3 (2i) [\sigma_z, [\sigma_z, \sigma_y]] + \dots \quad (35)
 \end{aligned}$$

$$\begin{aligned}
 &= \sigma_x + \alpha \left(-\frac{i}{2} \right) 2i\sigma_y - \frac{\alpha^2}{2} \left(-\frac{i}{2} \right)^2 (2i)^2 \sigma_x - \frac{\alpha^3}{6} \left(-\frac{i}{2} \right)^3 (2i)^2 [\sigma_z, \sigma_x] + \dots \\
 &= \sigma_x + \alpha\sigma_y - \frac{\alpha^2}{2}\sigma_x - \frac{\alpha^3}{6}\sigma_y + \dots \quad (36) \\
 &= \left(1 - \frac{\alpha^2}{2} + \dots \right) \sigma_x + \left(\alpha - \frac{\alpha^3}{6} + \dots \right) \sigma_y
 \end{aligned}$$

The series continues in the familiar pattern of $\cos \alpha$ and $\sin \alpha$ due to the pattern in (35), which leaves us:

$$e^{-\frac{i}{2}\alpha\sigma_z} \sigma_x e^{\frac{i}{2}\alpha\sigma_z} = \cos \alpha \sigma_x + \sin \alpha \sigma_y \quad (37)$$

The unitary matrix $\hat{U} = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}$, when transforming an operator/matrix, causes the transformations:

$$\sigma \rightarrow \hat{U} \sigma \hat{U}^\dagger = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} \sigma e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}$$

As we saw in (37), if $\mathbf{n} = (0, 0, 1) = z$, we got a rotation about the z axis. If we now imagine a different orthonormal coordinate system with unit vectors $\mathbf{n} = \mathbf{n}_z, \mathbf{n}_y, \mathbf{n}_x$, then the transformation $\hat{U}\sigma_{\mathbf{n}_x}\hat{U}^\dagger$ would look like:

$$\sigma_{\mathbf{n}_x} \rightarrow e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}\sigma_{\mathbf{n}_x}e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \cos\alpha\sigma_{\mathbf{n}_x} + \sin\alpha\sigma_{\mathbf{n}_y}$$

and rotates the spin basis around the axis \mathbf{n} .

c)

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^k}{k!}$$

Then we need to know what the different powers of $\sigma_{\mathbf{n}}$ are:

$$\begin{aligned} \sigma_{\mathbf{n}}^2 &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}^2 = \begin{pmatrix} \cos^2\theta + e^{-i\phi}e^{i\phi}\sin^2\theta & \cos\theta e^{-i\phi}\sin\theta - \cos\theta e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta\cos\theta - \cos\theta e^{i\phi}\sin\theta & e^{i\phi}\sin\theta e^{-i\phi}\sin\theta + \cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \mathbb{1} \end{aligned}$$

Thus, we see that $\sigma_{\mathbf{n}}$ follows the standard Pauli matrix identities: $\sigma_{\mathbf{n}}^{2k} = \mathbb{1}$ and $\sigma_{\mathbf{n}}^{2k+1} = \sigma_{\mathbf{n}}$ for $k = 0, 1, \dots$. This lets us rewrite our series as:

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^{2k+1}}{(2k+1)!}$$

Using that $(-i)^{2k} = (-1)^k$, $(-i)^{2k+1} = -i^{2k+1} = -i^{2k}i = (-1)^{k+1}i$, and the identities for the Pauli matrices, we get:

$$\begin{aligned} e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} &= \mathbb{1} \underbrace{\sum_{k=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{2k} (-1)^k}{(2k)!}}_{=\cos\frac{\alpha}{2}} - i\sigma_{\mathbf{n}} \underbrace{\sum_{k=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{2k+1} (-1)^k}{(2k+1)!}}_{=\sin\frac{\alpha}{2}} \\ e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} &= \cos\frac{\alpha}{2}\mathbb{1} - i\sin\frac{\alpha}{2}\sigma_{\mathbf{n}} \end{aligned}$$