

Problem set 3

3.1 Ladder operators in the Heisenberg picture

Alternative 1

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad \hat{U}(t, 0) = e^{-it\hat{H}/\hbar}$$

One way to find the time evolution of an operator in the heisenberg picture, is to transform them as follows:

$$\begin{aligned} \hat{a}(t) &= \hat{U}(t, 0)^\dagger \hat{a} \hat{U}(t, 0) = e^{it\hat{H}/\hbar} \hat{a} e^{-it\hat{H}/\hbar} \\ \hat{a}^\dagger(t) &= \hat{U}(t, 0)^\dagger \hat{a}^\dagger \hat{U}(t, 0) = e^{it\hat{H}/\hbar} \hat{a}^\dagger e^{-it\hat{H}/\hbar} \end{aligned}$$

We know that

$$e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}} = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$$

We need the commutators

$$\begin{aligned} [\hat{H}, \hat{a}] &= \hbar\omega \left[\hat{a}^\dagger \hat{a} + \frac{1}{2}, \hat{a} \right] = \hbar\omega [\hat{a}^\dagger \hat{a}, \hat{a}] = \hbar\omega \left([\hat{a}^\dagger, \hat{a}] \hat{a} + \hat{a}^\dagger [\hat{a}, \hat{a}] \right) = -\hbar\omega \hat{a} \\ [\hat{H}, \hat{a}^\dagger] &= \hbar\omega \left[\hat{a}^\dagger \hat{a} + \frac{1}{2}, \hat{a}^\dagger \right] = \hbar\omega [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hbar\omega \left([\hat{a}^\dagger, \hat{a}^\dagger] \hat{a} + \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] \right) = \hbar\omega \hat{a}^\dagger \end{aligned}$$

in order to get:

$$\begin{aligned} e^{it\hat{H}/\hbar} \hat{a} e^{-it\hat{H}/\hbar} &= \hat{a} + \frac{it}{\hbar} [\hat{H}, \hat{a}] + \frac{1}{2!} \left(\frac{it}{\hbar} \right)^2 [\hat{H}, [\hat{H}, \hat{a}]] + \dots \\ &= \hat{a} - it\omega \hat{a} + \frac{1}{2!} \left(\frac{it}{\hbar} \right)^2 (-\hbar\omega)^2 \hat{a} + \dots \\ &= \hat{a} \left(1 + (-it\omega) + \frac{(-it\omega)^2}{2!} + \dots \right) \\ &= \hat{a} e^{-it\omega} \end{aligned}$$

and

$$\begin{aligned} e^{it\hat{H}/\hbar} \hat{a}^\dagger e^{-it\hat{H}/\hbar} &= \hat{a}^\dagger + \frac{it}{\hbar} [\hat{H}, \hat{a}^\dagger] + \frac{1}{2!} \left(\frac{it}{\hbar} \right)^2 [\hat{H}, [\hat{H}, \hat{a}^\dagger]] + \dots \\ &= \hat{a}^\dagger + it\omega \hat{a}^\dagger + \frac{1}{2!} \left(\frac{it}{\hbar} \right)^2 (\hbar\omega)^2 \hat{a}^\dagger + \dots \\ &= \hat{a}^\dagger \left(1 + (it\omega) + \frac{(it\omega)^2}{2!} + \dots \right) \\ &= \hat{a}^\dagger e^{it\omega} \end{aligned}$$

So we have:

$$\hat{a}(t) = \hat{a} e^{-it\omega}, \quad \hat{a}^\dagger(t) = \hat{a}^\dagger e^{it\omega}$$

Alternative 2 Another option is to use the Heisenberg equation of motion:

$$\dot{\hat{a}} = \frac{i}{\hbar} [H, \hat{a}] = i\omega [\hat{a}^\dagger \hat{a}, \hat{a}] = -i\omega \hat{a} \quad (1)$$

The solution to this equation is the same if a is an operator as if it was a number, and we get

$$\hat{a}(t) = \hat{a} e^{-it\omega},$$

and similarly for \hat{a}^\dagger .

3.2 Displacement operators in phase space

We have the Campbell-Baker-Hausdorff expansion: $e^{\lambda\hat{A}}e^{\lambda\hat{B}} = e^{\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}[\hat{A},\hat{B}]+\dots}$. Let's examine the commutators, with $\hat{A} = z_a\hat{a}^\dagger - z_a^*\hat{a}$, $\hat{B} = z_b\hat{a}^\dagger - z_b^*\hat{a}$.

$$\begin{aligned} [\hat{A}, \hat{B}] &= [z_a\hat{a}^\dagger - z_a^*\hat{a}, z_b\hat{a}^\dagger - z_b^*\hat{a}] = \underbrace{[z_a\hat{a}^\dagger, z_b\hat{a}^\dagger]}_{=0} + [z_a\hat{a}^\dagger, -z_b^*\hat{a}] + [-z_a^*\hat{a}, z_b\hat{a}^\dagger] + \underbrace{[-z_a^*\hat{a}, -z_b^*\hat{a}]}_{=0} \\ &= z_a z_b^* - z_a^* z_b = \underbrace{\text{Re}(z_a z_b^*) - \text{Re}(z_a^* z_b)}_{=0} + \underbrace{i \text{Im}(z_a z_b^*) - i \text{Im}(z_a^* z_b)}_{=i \text{Im}(z_a z_b^*) - i \text{Im}([z_a z_b^*]^*)} = 2i \text{Im}(z_a z_b^*) \\ [\hat{B}, \hat{A}] &= -[\hat{A}, \hat{B}] = -2i \text{Im}(z_a z_b^*) \end{aligned}$$

These are complex numbers (and not operators), which means higher order terms vanish, as everything commutes with numbers. We're ready to examine the problem:

$$\begin{aligned} \hat{D}(z_a) \hat{D}(z_b) &= e^{\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}[\hat{A},\hat{B}]} \\ &\stackrel{\lambda=1}{=} e^{z_a\hat{a}^\dagger - z_a^*\hat{a} + z_b\hat{a}^\dagger - z_b^*\hat{a} + \frac{1}{2}(z_a z_b^* - z_a^* z_b)} \\ &= e^{(z_a+z_b)\hat{a}^\dagger - (z_a^*+z_b^*)\hat{a} + i \text{Im}(z_a z_b^*)} \end{aligned}$$

If we change the order, the only thing that changes is the order in the commutator:

$$\hat{D}(z_b) \hat{D}(z_a) = e^{(z_a+z_b)\hat{a}^\dagger - (z_a^*+z_b^*)\hat{a} - i \text{Im}(z_a z_b^*)}$$

From this we see that

$$\hat{D}(z_a) \hat{D}(z_b) = e^{2i \text{Im}(z_a z_b^*)} \hat{D}(z_b) \hat{D}(z_a)$$

This makes: $\alpha(z_a, z_b) = 2 \text{Im}(z_a z_b^*)$, and the condition for the displacements to commute, is $\text{Im}(z_a z_b^*) = 0$. Written out, this becomes:

$$\begin{aligned} \frac{1}{2m\hbar\omega} \text{Im} \left[(m\omega x_c^a + ip_c^a) (m\omega x_c^b - ip_c^b) \right] &= 0 \\ \text{Im} \left[(m\omega)^2 x_c^a x_c^b - im\omega x_c^a p_c^b + im\omega p_c^a x_c^b + p_c^a p_c^b \right] &= 0 \\ -m\omega x_c^a p_c^b + m\omega p_c^a x_c^b &= 0 \end{aligned}$$

The resulting condition is:

$$\frac{x_c^a}{p_c^a} = \frac{x_c^b}{p_c^b}$$

Geometrically it means that $z_b = cz_a$ with $c \in \mathbb{R}$ a real constant, which means that z_a and z_b lie on the same line passing through the origin in the complex plane.

3.3 Eigenvectors for \hat{a}^\dagger ?

We assume $\hat{a}^\dagger|\bar{z}\rangle = \bar{z}|\bar{z}\rangle$ and expand in the basis $|n\rangle$.

Starting with the left hand side:

$$\begin{aligned}\hat{a}^\dagger|\bar{z}\rangle &= \hat{a}^\dagger \sum_{n=0}^{\infty} |n\rangle \langle n|\bar{z}\rangle \\ &= \sum_{n=0}^{\infty} \langle n|\bar{z}\rangle \sqrt{n+1} |n+1\rangle \\ &= \sum_{n=1}^{\infty} \langle n-1|\bar{z}\rangle \sqrt{n} |n\rangle\end{aligned}$$

Then the right:

$$\bar{z}|\bar{z}\rangle = \bar{z} \sum_{n=0}^{\infty} \langle n|\bar{z}\rangle |n\rangle$$

Equating:

$$\sum_{n=1}^{\infty} \langle n-1|\bar{z}\rangle \sqrt{n} |n\rangle = \bar{z} \sum_{n=0}^{\infty} \langle n|\bar{z}\rangle |n\rangle$$

This gives for all n:

$$\langle n|\bar{z}\rangle = \langle n-1|\bar{z}\rangle \frac{\sqrt{n}}{\bar{z}}$$

and

$$\langle 0|\bar{z}\rangle = 0$$

as there is no corresponding term. This means that $\langle 1|\bar{z}\rangle = \langle 0|\bar{z}\rangle \frac{\sqrt{1}}{\bar{z}} = 0$, and thus all coefficients are 0 by induction. As the sum of coefficients that are exactly 0 is still 0, we cannot satisfy $\sum_{n=0}^{\infty} |\langle n|\bar{z}\rangle|^2 = 1$, and therefore not normalize it.

3.4 A driven harmonic oscillator (Exam 2010)

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \hbar\lambda \left(\hat{a}^\dagger e^{-i\omega t} + \hat{a} e^{i\omega t} \right), \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{x} = \frac{1}{2} (\hat{a} + \hat{a}^\dagger), \quad \hat{p} = -\frac{i}{2} (\hat{a} - \hat{a}^\dagger)$$

a) We're given the Heisenberg equation of motion, which can be used on operators in order to get a differential equation that determines the time evolution of the operator

$$\frac{d}{dt} \hat{A} = \frac{i}{\hbar} [\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t}$$

Note that $\frac{\partial \hat{A}}{\partial t}$ is the explicit time derivative of the operator in the Schrödinger picture. The ladder operators have no explicit time dependence in the Schrödinger picture, and thus $\frac{\partial \hat{a}}{\partial t} = 0$.

Using

$$[\hat{H}, \hat{a}] = -\hbar\omega_0 \hat{a} - \hbar\lambda e^{-i\omega t} \Rightarrow \frac{d^2 \hat{a}}{dt^2} = -i \frac{d}{dt} (\omega_0 \hat{a} + \lambda e^{-i\omega t})$$

we have that

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{a}] = -i (\omega_0 \hat{a} + \lambda e^{-i\omega t})$$

Taking the derivative we get

$$\begin{aligned} \frac{d^2 \hat{a}}{dt^2} &= -i\omega_0 \frac{d\hat{a}}{dt} - \lambda\omega e^{-i\omega t} \\ &= -\omega_0^2 \hat{a} - (\omega_0 + \omega) \lambda e^{-i\omega t} \end{aligned}$$

Conjugating we get

$$\frac{d^2 \hat{a}^\dagger}{dt^2} = -\omega_0^2 \hat{a}^\dagger - (\omega_0 + \omega) \lambda e^{i\omega t}.$$

Then,

$$\begin{aligned} \frac{d^2 \hat{x}}{dt^2} &= \frac{1}{2} \left(\frac{d^2 \hat{a}}{dt^2} + \frac{d^2 \hat{a}^\dagger}{dt^2} \right) = \frac{1}{2} \left(-\omega_0^2 \hat{a} - (\omega_0 + \omega) \lambda e^{-i\omega t} - \omega_0^2 \hat{a}^\dagger - (\omega_0 + \omega) \lambda e^{i\omega t} \right) \\ &= \frac{1}{2} \left(-\omega_0^2 (\hat{a} + \hat{a}^\dagger) - \lambda (\omega_0 + \omega) (e^{i\omega t} + e^{-i\omega t}) \right) \\ &= -\omega_0^2 \hat{x} - \lambda (\omega_0 + \omega) \frac{1}{2} (e^{i\omega t} + e^{-i\omega t}) \\ &= -\omega_0^2 \hat{x} - \lambda (\omega_0 + \omega) \cos \omega t \end{aligned}$$

This is indeed on the form:

$$\frac{d^2 \hat{x}}{dt^2} + \omega_0^2 \hat{x} = C \cos \omega t, \quad C = -\lambda (\omega_0 + \omega)$$

b) Using that $\hat{H}_T|\psi_T(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi_T(t)\rangle$, we get

$$\begin{aligned}\hat{H}_T|\psi_T(t)\rangle &= i\hbar\frac{\partial}{\partial t}\hat{T}(t)|\psi(t)\rangle \\ &= i\hbar\frac{\partial\hat{T}}{\partial t}|\psi(t)\rangle + i\hbar\hat{T}(t)\frac{\partial}{\partial t}|\psi(t)\rangle \\ &= i\hbar\frac{\partial\hat{T}}{\partial t}|\psi(t)\rangle + \hat{T}(t)\hat{H}(t)|\psi(t)\rangle \\ \hat{H}_T|\psi_T(t)\rangle &= i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger|\psi_T(t)\rangle + \hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger|\psi_T(t)\rangle \\ \Rightarrow \hat{H}_T &= \hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger + i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger\end{aligned}$$

Then we use that: $\hat{T}(t) = e^{i\omega t \hat{a}^\dagger \hat{a}} \Rightarrow \frac{\partial\hat{T}}{\partial t} = i\omega \hat{a}^\dagger \hat{a} \hat{T}(t)$ and $\hat{T}(t)^\dagger = e^{-i\omega t \hat{a}^\dagger \hat{a}}$, the last one is due to $\hat{a}^\dagger \hat{a} = \hat{N}$ which is hermitian. Then

$$i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger = i\hbar i\omega \hat{a}^\dagger \hat{a} \hat{T}(t)\hat{T}(t)^\dagger = -\hbar\omega \hat{a}^\dagger \hat{a}$$

We define the two terms in the Hamiltonian as

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \hbar\lambda \left(\hat{a}^\dagger e^{-i\omega t} + \hat{a} e^{i\omega t} \right) \equiv \hat{H}_0 + \hat{H}_1$$

We see that $[\hat{H}_0, \hat{T}(t)] = 0$ which means that

$$\hat{T}(t)\hat{H}_0\hat{T}(t)^\dagger = \hat{H}_0$$

We have that

$$e^{i\omega t \hat{a}^\dagger \hat{a}} \hat{H}_1 e^{-i\omega t \hat{a}^\dagger \hat{a}} = \hbar\lambda \left(e^{i\omega t \hat{a}^\dagger \hat{a}} \hat{a}^\dagger e^{-i\omega t \hat{a}^\dagger \hat{a}} e^{-i\omega t} + e^{i\omega t \hat{a}^\dagger \hat{a}} \hat{a} e^{-i\omega t \hat{a}^\dagger \hat{a}} e^{i\omega t} \right)$$

Applying $e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}} = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$, with the commutators:

$$\begin{aligned}[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] &= [\hat{a}^\dagger, \hat{a}^\dagger] \hat{a} + \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] \\ &= \hat{a}^\dagger \\ \Rightarrow [\hat{a}^\dagger \hat{a}, [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger]] &= [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] \\ &= \hat{a}^\dagger \\ [\hat{a}^\dagger \hat{a}, \hat{a}] &= [\hat{a}^\dagger, \hat{a}] \hat{a} + \hat{a}^\dagger [\hat{a}, \hat{a}] \\ &= -\hat{a} \\ \Rightarrow [\hat{a}^\dagger \hat{a}, [\hat{a}^\dagger \hat{a}, \hat{a}]] &= \hat{a}\end{aligned}$$

we get

$$\begin{aligned}e^{i\omega t \hat{a}^\dagger \hat{a}} \hat{H}_1 e^{-i\omega t \hat{a}^\dagger \hat{a}} &= \hbar\lambda \left(\hat{a}^\dagger \left(\sum_{n=0}^{\infty} \frac{(i\omega t)^n}{n!} \right) e^{-i\omega t} + \hat{a} \left(\sum_{n=0}^{\infty} \frac{(-i\omega t)^n}{n!} \right) e^{i\omega t} \right) \\ &= \hbar\lambda \left(\hat{a}^\dagger e^{i\omega t} e^{-i\omega t} + \hat{a} e^{-i\omega t} e^{i\omega t} \right) \\ &= \hbar\lambda \left(\hat{a}^\dagger + \hat{a} \right).\end{aligned}$$

Then we find

$$\begin{aligned}\hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger &= \hat{H}_0 + \hbar\lambda (\hat{a}^\dagger + \hat{a}) \\ &= \hbar\omega_0 \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} \right) + \hbar\lambda (\hat{a}^\dagger + \hat{a})\end{aligned}$$

The full Hamiltonian is:

$$\begin{aligned}\hat{H}_T &= \hat{T}(t)\hat{H}(t)\hat{T}(t)^\dagger + i\hbar\frac{\partial\hat{T}}{\partial t}\hat{T}(t)^\dagger \\ &= \hbar\omega_0 \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} \right) + \hbar\lambda (\hat{a}^\dagger + \hat{a}) - \hbar\omega\hat{a}^\dagger\hat{a} \\ &= \hbar(\omega_0 - \omega)\hat{a}^\dagger\hat{a} + \hbar\lambda (\hat{a}^\dagger + \hat{a}) + \frac{1}{2}\hbar\omega_0\end{aligned}$$

- c) If $|\psi(t)\rangle$ evolves as a coherent state, it satisfies: $\hat{a}|\psi(t)\rangle = \psi(t)|\psi(t)\rangle$. Let's note that the hamiltonian in the Schrödinger picture has an explicit time dependence, and in order to get the time evolution operator, we would want it to be time-independent. We know that \hat{H}_T is time independent and generates the time dependence through $\hat{U}_T(t) = e^{-it\hat{H}_T/\hbar}$. Thus, in order to get the time dependence in the original picture, we need to write $\hat{U}(t)$ in terms of $\hat{U}_T(t)$:

$$\begin{aligned}|\psi_T(t)\rangle &= \hat{T}(t)|\psi(t)\rangle = \hat{T}(t)\hat{U}(t)|\psi(0)\rangle \\ |\psi_T(t)\rangle &= \hat{U}_T(t)|\psi_T(0)\rangle \\ \Rightarrow \hat{U}_T(t) &= \hat{T}(t)\hat{U}(t) \Rightarrow \hat{U}(t) = \hat{T}(t)^\dagger\hat{U}_T(t)\end{aligned}$$

If $|\psi(t)\rangle$ evolves as a coherent state, then it is an eigenstate of \hat{a} :

$$\hat{a}|\psi(t)\rangle = \hat{a}\hat{U}(t)|0\rangle = \hat{U}(t)\hat{U}(t)^\dagger\hat{a}\hat{U}(t)|0\rangle = \hat{U}(t)\hat{U}_T(t)^\dagger\hat{T}(t)\hat{a}\hat{T}(t)^\dagger\hat{U}_T(t)|0\rangle$$

From exercise b), we got that $\hat{T}(t)\hat{a}\hat{T}(t)^\dagger = \hat{a}e^{-i\omega t}$, which leaves us with:

$$\hat{a}|\psi(t)\rangle = e^{-i\omega t}\hat{U}(t)\hat{U}_T(t)^\dagger\hat{a}\hat{U}_T(t)|0\rangle = e^{-i\omega t}\hat{U}(t)e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar}|0\rangle \quad (2)$$

Expanding $e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar}$ yields:

$$e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar} = \hat{a} + \frac{it}{\hbar} [\hat{H}_T, \hat{a}] + \left(\frac{it}{\hbar}\right)^2 [\hat{H}_T, [\hat{H}_T, \hat{a}]] + \dots$$

The commutators are:

$$\begin{aligned}[\hat{H}_T, \hat{a}] &= \left[\hbar(\omega_0 - \omega)\hat{a}^\dagger\hat{a} + \hbar\lambda (\hat{a}^\dagger + \hat{a}) + \frac{1}{2}\hbar\omega_0, \hat{a} \right] \\ &= \left[\hbar(\omega_0 - \omega)\hat{a}^\dagger\hat{a}, \hat{a} \right] + \left[\hbar\lambda (\hat{a}^\dagger + \hat{a}), \hat{a} \right] + \underbrace{\left[\frac{1}{2}\hbar\omega_0, \hat{a} \right]}_{=0} \\ &= \hbar(\omega_0 - \omega) [\hat{a}^\dagger\hat{a}, \hat{a}] + \hbar\lambda \left([\hat{a}^\dagger, \hat{a}] + [\hat{a}, \hat{a}] \right) \\ &= -\hbar(\omega_0 - \omega)\hat{a} - \hbar\lambda \\ &= \hbar(\omega - \omega_0)\hat{a} - \hbar\lambda\end{aligned}$$

$$\begin{aligned}
 [\hat{H}_T, [\hat{H}_T, \hat{a}]] &= [\hat{H}_T, \hbar(\omega - \omega_0)\hat{a} - \hbar\lambda] \\
 &= \hbar(\omega_0 - \omega) [\hat{H}_T, \hat{a}] \\
 [\hat{H}_T, [\hat{H}_T, [\hat{H}_T, \hat{a}]]] &= \hbar(\omega - \omega_0) [\hat{H}_T, [\hat{H}_T, \hat{a}]] \\
 &= [\hbar(\omega - \omega_0)]^2 [\hat{H}_T, \hat{a}]
 \end{aligned}$$

We see a pattern emerging:

$$\begin{aligned}
 e^{it\hat{H}_T/\hbar}\hat{a}e^{-it\hat{H}_T/\hbar} &= \hat{a} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar}\right)^n [\hbar(\omega - \omega_0)]^{n-1} [\hat{H}_T, \hat{a}] \\
 &= \hat{a} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar}\right)^n [\hbar(\omega - \omega_0)]^{n-1} (\hbar(\omega - \omega_0)\hat{a} - \hbar\lambda) \\
 &= \hat{a} + \hat{a} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar}\right)^n [\hbar(\omega - \omega_0)]^n - \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar}\right)^n [\hbar(\omega - \omega_0)]^{n-1} \hbar\lambda \\
 &= \hat{a} \sum_{n=0}^{\infty} \frac{1}{n!} [it(\omega - \omega_0)]^n - \sum_{n=1}^{\infty} \frac{1}{n!} (it)^n [(\omega - \omega_0)]^{n-1} \lambda \\
 &= \hat{a} \sum_{n=0}^{\infty} \frac{1}{n!} [it(\omega - \omega_0)]^n - \frac{\lambda}{(\omega - \omega_0)} \sum_{n=1}^{\infty} \frac{1}{n!} [it(\omega - \omega_0)]^n \\
 &= \hat{a}e^{it(\omega - \omega_0)} - \frac{\lambda}{(\omega - \omega_0)} (e^{it(\omega - \omega_0)} - 1)
 \end{aligned}$$

Inserting back into (2) gives:

$$\hat{a}|\psi(t)\rangle = e^{-i\omega t}\hat{U}(t) \left(\hat{a}e^{it(\omega - \omega_0)} - \frac{\lambda}{(\omega - \omega_0)} (e^{it(\omega - \omega_0)} - 1) \right) |0\rangle$$

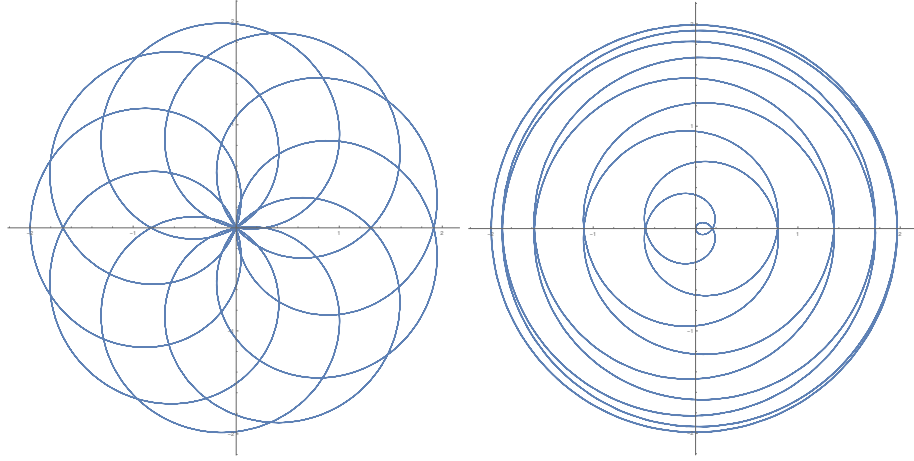
We note that $\hat{a}|0\rangle = 0$:

$$\begin{aligned}
 \hat{a}|\psi(t)\rangle &= e^{-i\omega t}\hat{U}(t) \frac{\lambda}{(\omega - \omega_0)} (1 - e^{it(\omega - \omega_0)}) |0\rangle \\
 &= e^{-i\omega t} \frac{\lambda}{(\omega - \omega_0)} (1 - e^{it(\omega - \omega_0)}) \hat{U}(t)|0\rangle \\
 &= e^{-i\omega t} \frac{\lambda}{(\omega - \omega_0)} (1 - e^{it(\omega - \omega_0)}) |\psi(t)\rangle \\
 &= \frac{\lambda}{(\omega - \omega_0)} (e^{-i\omega t} - e^{-it\omega_0}) |\psi(t)\rangle \\
 &\equiv z(t)|z(t)\rangle
 \end{aligned}$$

We see that the number $z(t)$ is the difference of two complex numbers which have the same constant absolute value while rotating with different frequencies. It will be 0 when the two are in phase and maximal when they are out of phase. To get some better understanding we can rewrite

$$e^{-i\omega t} - e^{-it\omega_0} = 2i \sin \frac{\omega_0 - \omega}{2} t e^{i\frac{\omega_0 + \omega}{2} t}$$

This means that $z(t)$ will follow a spiral path with the phase increasing with a frequency equal to $\frac{\omega_0 + \omega}{2}$ while the amplitude changes with the frequency $\frac{\omega_0 - \omega}{2}$. Here are two examples (Left: $\omega_0 = 1, \omega = 0.1$, Right: $\omega_0 = 1, \omega = 0.9$)



The real part is given as:

$$x(t) = \frac{1}{2} (z(t) + z(t)^*) = \frac{1}{2} \frac{\lambda}{(\omega - \omega_0)} (e^{-i\omega t} - e^{-i\omega_0 t} + e^{i\omega t} - e^{i\omega_0 t}) = \frac{\lambda}{(\omega - \omega_0)} (\cos \omega t - \cos \omega_0 t)$$

Finding the equation of motion:

$$\frac{d^2 x}{dt^2} = \frac{\lambda}{(\omega - \omega_0)} (-\omega^2 \cos \omega t + \omega_0^2 \cos \omega_0 t)$$

We want this to be equal to $C \cos \omega t - \omega_0^2 \frac{\lambda}{(\omega - \omega_0)} (\cos \omega t - \cos \omega_0 t)$. The trick is then to add in 0:

$$\begin{aligned} \frac{d^2 x}{dt^2} &= \frac{\lambda}{(\omega - \omega_0)} (-\omega^2 \cos \omega t + \omega_0^2 (\cos \omega_0 t + \cos \omega t - \cos \omega t)) \\ &= \frac{\lambda}{(\omega - \omega_0)} ((-\omega^2 + \omega_0^2) \cos \omega t + \omega_0^2 (\cos \omega_0 t - \cos \omega t)) \\ &= \frac{\lambda}{(\omega - \omega_0)} (-\omega^2 + \omega_0^2) \cos \omega t - \omega_0^2 x = -\lambda (\omega + \omega_0) \cos \omega t - \omega_0^2 x \end{aligned}$$

Thus, we see it satisfies the equation with $C = -\lambda (\omega + \omega_0)$.