Solutions to problem set 6

6.1 Entanglement and measurements

The problem lies in the sentences ”Following this measurement, suppose that the x-component of the spin of particle 1 is measured. It will be found to have the value $\hbar/2$ or $-\hbar/2$, and the z-component of particle 1s spin will no longer have a definite value. Also, because the system has zero total angular momentum, the spin of particle 2 will then have x-component $-\hbar/2$ or $\hbar/2$, and its z-component will not have a definite value.” It seems that it is assumed that first the spin is measured along z and then subsequently along x. But the first measurement along z will collapse the wavefunction, destroying all entanglement between the two particles. Measuring along x after that will give random uncorrelated results on the two particles, and not the perfect anticorrelation as stated. The appeal to ”zero total angular momentum” is not relevant, as the interaction with the measuring device can change the angular momentum, as it does even when considering measuring the spin along different axes for a single particle. The text would be fine if instead of ”Following this measurement, suppose...” we write ”Supose instead...”. This would mean that we can choose to measure either along z or x (but not both), and in both cases will we be able to deduce the corresponding spin component of the other particle. This component must then correspond to an element of reality according to EPR, and this is what they wanted to explain.

6.2 Hidden variables for a single spin-$\frac{1}{2}$

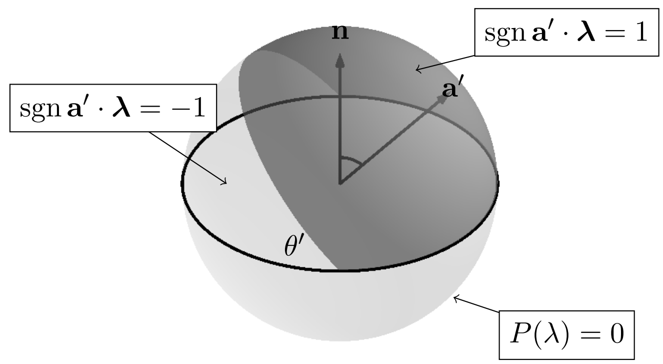
a) We know that

$$\langle\psi_{\mathbf{n}}|\boldsymbol{\sigma}|\psi_{\mathbf{n}}\rangle = \mathbf{n}$$

Thus,

$$\langle\psi_{\mathbf{n}}|\boldsymbol{\sigma}_{\mathbf{a}}|\psi_{\mathbf{n}}\rangle = \langle\psi_{\mathbf{n}}|\mathbf{a}\cdot\boldsymbol{\sigma}|\psi_{\mathbf{n}}\rangle = \mathbf{a}\cdot\langle\psi_{\mathbf{n}}|\boldsymbol{\sigma}|\psi_{\mathbf{n}}\rangle = \mathbf{a}\cdot\mathbf{n} = \cos\theta$$

b) In the figure, the surface represents the possible values of the unit vector $\boldsymbol{\lambda}$. In the upper hemisphere the distribution function $P(\lambda)$ is constant while it is zero on the bottom hemisphere.



The fraction of the upper hemisphere where sgn $\mathbf{a}' \cdot \boldsymbol{\lambda} = -1$ is $\frac{\theta'}{\pi}$ while the fraction with sgn $\mathbf{a}' \cdot \boldsymbol{\lambda} = 1$ is $1 - \frac{\theta'}{\pi}$. This gives

$$\langle A_{\mathbf{a}} \rangle = \int d\lambda P(\lambda) A(\mathbf{a}, \boldsymbol{\lambda}) = \frac{\theta'}{\pi}(-1) + (1 - \frac{\theta'}{\pi})(+1) = 1 - \frac{2\theta'}{\pi}.$$

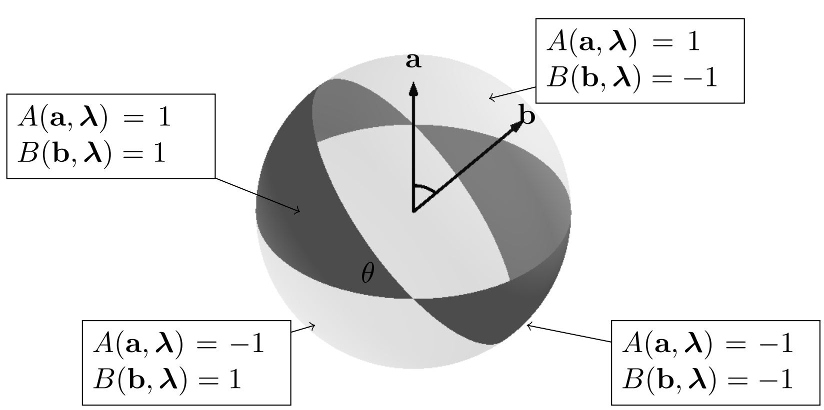
c) We must choose $\mathbf{a}'$ so that $1 - \frac{2\theta'}{\pi} = \cos\theta$.

6.3 Hidden variables for anticorrelation of a pair of spin-$\frac{1}{2}$

a)

$$\begin{aligned}\langle \psi | \sigma^A_{\mathbf{a}} \sigma^B_{\mathbf{a}} | \psi \rangle &= \frac{1}{2}\left[\langle \uparrow\downarrow | \sigma^A_{\mathbf{a}} \sigma^B_{\mathbf{a}} | \uparrow\downarrow \rangle - \langle \uparrow\downarrow | \sigma^A_{\mathbf{a}} \sigma^B_{\mathbf{a}} | \downarrow\uparrow \rangle - \langle \downarrow\uparrow | \sigma^A_{\mathbf{a}} \sigma^B_{\mathbf{a}} | \uparrow\downarrow \rangle + \langle \downarrow\uparrow | \sigma^A_{\mathbf{a}} \sigma^B_{\mathbf{a}} | \downarrow\uparrow \rangle \right] \\ &= \frac{1}{2}\left[a_z(-a_z) - (a_x - ia_y)(a_x + ia_y) - (a_x + ia_y)(a_x - ia_y) + (-a_z)a_z \right] = -\mathbf{a} \cdot \mathbf{a} = -1.\end{aligned}$$

b) In the figure we have shown in dark the areas where $A_{\mathbf{a}} B_{\mathbf{b}} = 1$ and light the areas where $A_{\mathbf{a}} B_{\mathbf{b}} = -1$



Since λ is uniformly distributed over the sphere, the probabilites for each area are propotional to the fraction of the sphere they cover. The dark areas cover a fraction θ/π while the light areas cover $(\pi - \theta)/\pi$ of the sphere. Then we get

$$\langle A_{\mathbf{a}} B_{\mathbf{b}} \rangle = 1 \cdot \theta/\pi + (-1) \cdot (\pi - \theta)/\pi = -1 + \frac{2}{\pi}\theta.$$

6.4 Greenberger-Horne-Zeilinger (GHZ) version of Bell's theorem

a) We have

$$[\Sigma_A, \Sigma_B] = \sigma_x^A \sigma_y^B \sigma_y^C \sigma_y^A \sigma_x^B \sigma_y^C - \sigma_y^A \sigma_x^B \sigma_y^C \sigma_x^A \sigma_y^B \sigma_y^C$$

In the last term we use the fact that operators on different particles commute, while for the same particle, two different Pauli matrices anticommute ($\sigma_x \sigma_y = -\sigma_y \sigma_x$ etc.) to get

$$\sigma_y^A \sigma_x^B \sigma_y^C \sigma_x^A \sigma_y^B \sigma_y^C = \sigma_y^A \sigma_x^A \sigma_x^B \sigma_y^B \sigma_y^C \sigma_y^C = \sigma_x^A \sigma_y^A \sigma_y^B \sigma_x^B \sigma_y^C \sigma_y^C$$

since we have two anticommuting pairs. This is precisely the first term, and we have shown that $[\Sigma_A, \Sigma_B] = 0$. The other commutators are shown to be 0 in a similar way.

b) We have

$$\sigma_x|\uparrow\rangle = |\downarrow\rangle, \qquad \sigma_y|\uparrow\rangle = i|\downarrow\rangle$$
$$\sigma_x|\downarrow\rangle = |\uparrow\rangle, \qquad \sigma_y|\downarrow\rangle = -i|\uparrow\rangle$$

and get

$$\Sigma_A|\psi\rangle = \frac{1}{\sqrt{2}} \sigma_x^A \sigma_y^B \sigma_y^C \left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right) = \frac{1}{\sqrt{2}} \left(-|\downarrow\downarrow\downarrow\rangle + |\uparrow\uparrow\uparrow\rangle\right) = |\psi\rangle \tag{1}$$

and similar or the other two. If all spins are measured, with two measured along the $y$direction and one along the x-direction, the product of all three results will always be the eigenvalue of the corresponding Σ_i, which as we have shown is $+1$. Since the eigenvalues of each spin operator is $+1$ if te spin is up and -1 if the spin is down, it means that we will always get an even number of spins down along their chosen axis.

c)

$$\Sigma|\psi\rangle = \frac{1}{\sqrt{2}} \sigma_x^A \sigma_x^B \sigma_x^C \left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right) = \frac{1}{\sqrt{2}} \left(|\downarrow\downarrow\downarrow\rangle - |\uparrow\uparrow\uparrow\rangle\right) = -|\psi\rangle \tag{2}$$

d) We have the three equations

$$A_x B_y C_y = 1 \qquad A_y B_x C_y = 1 \qquad A_y B_y C_x = 1 \tag{3}$$

Take the product of all three:

$$A_x B_y C_y A_y B_x C_y A_y B_y C_x = A_x A_y^2 B_x B_y^2 C_x C_y^2 = A_x B_x C_x = 1$$

which is not consistent with

$$A_x B_x C_x = -1 \tag{4}$$

6.5 Tsirelson's bound

a)

$$\begin{aligned} S^2 &= A^2B^2 + A^2BB' + AA'B^2 - AA'BB' + A^2B'B + A^2B'^2 + AA'B'B - AA'B'^2 \\ &\quad + A'AB^2 + A'ABB' + A'^2B^2 - A'^2BB' - A'AB'B - A'AB'^2 - A'^2B'B + A'^2B'^2 \\ &= 4 - AA'BB' + AA'B'B + A'ABB' - A'AB'B \\ &= 4 - [A, A'][B, B'] \end{aligned}$$

b) We have that

$$\|N\| = \sup_{|\psi\rangle} \frac{\|N|\psi\rangle\|}{\||\psi\rangle\|}$$

which implies that for any $|\psi\rangle$ is

$$\|N\| \geq \frac{\|N|\psi\rangle\|}{\||\psi\rangle\|}$$

or

$$\||\psi\rangle\| \geq \frac{\|N|\psi\rangle\|}{\|N\|}.$$

This means that

$$\frac{\|MN|\psi\rangle\|}{\||\psi\rangle\|} \leq \frac{\|MN|\psi\rangle\|}{\|N|\psi\rangle\|} \|N\|$$

Then

$$\|MN\| \leq \|N\| \sup_{|\psi\rangle} \frac{\|MN|\psi\rangle\|}{\|N|\psi\rangle\|} = \|N\| \sup_{|\phi\rangle} \frac{\|M|\phi\rangle\|}{\||\phi\rangle\|} = \|M\| \, \|N\|$$

where $|\phi\rangle = N|\psi\rangle$.

The triangle inequality for elements in a vector space reads

$$\|(M+N)|\psi\rangle\| \leq \|M|\psi\rangle\| + \|N|\psi\rangle\|$$

Taking the supremum on both sides we have

$$\|M+N\| \leq \sup_{|\psi\rangle} \frac{\|(M+N)|\psi\rangle\|}{\||\psi\rangle\|} \leq \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} + \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} = \|M\| + \|N\|$$

c) We have

$$\|[M, N]\| = \|MN - NM\| \leq \|MN\| + \|NM\| \leq 2\,\|M\|\,\|N\|\,.$$

Also it is clear that $\|A\| = \|B\| = \|A'\| = \|B'\| = 1$. This means

$$\left\|S^2\right\| \leq 4 + 4\,\|A\|\,\left\|A'\right\|\,\|B\|\,\left\|B'\right\| = 8$$

which gives $\|S\| \leq 2\sqrt{2}$.