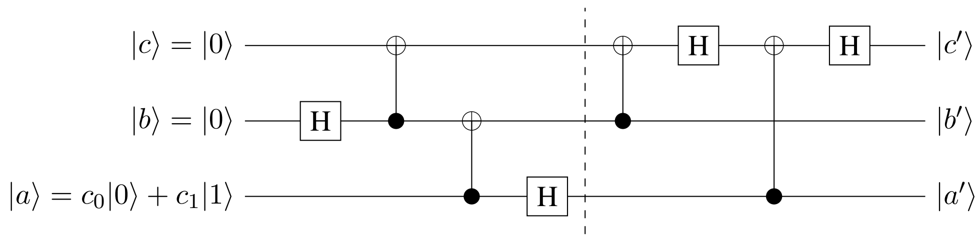
Problem set 9

9.1 Quantum gates for teleportation

We consider the following quantum circuit (ignore the vertical dashed line for the moment)



a) Confirm that the final state is a product state despite the entanglement created in the middle of the circuit and that $|c'\rangle = |a\rangle$ so that the state of the lowest qubit is teleported to the upper qubit.

b) We now measure the a and b qubits at the vertical dashed line and perform the two remaining CNOT gates based on the outcomes of these measurements. That is, they are local at qubit c. Check that we still find $|c'\rangle = |a\rangle$ at the end.

9.2 Quantum cloning of orthogonal states

The no-cloning theorem tells us that it is not possible to make a copy of an arbitrary initial state. However, if we know that the states we have to copy are not general, but selected from a set of orthogonal states, we can find a way to copy them.

a) Given two orthgonal states $|\psi\rangle$ and $|\phi\rangle$ for a single qubit, design a quantum circuit with two input qubits with the following properties. If the first qubit is in the state to be copied, which is always either $|\psi\rangle$ or $|\phi\rangle$ and the second qubit is in a standard state $|0\rangle$, the output is $|\psi\rangle|\psi\rangle$ or $|\phi\rangle|\phi\rangle$ depending on whether $|\psi\rangle$ or $|\phi\rangle$ was input on the first qubit (the circuit is not general, it will depend on which states $|\psi\rangle$ and $|\phi\rangle$ are used). Assume that you can use as elementary gates in the circuit all single qubit gates and CNOT.

b) Assume for simplicity that the two orthogonal states are $|0\rangle$ and $|1\rangle$. What is the output of the circuit if the input is the superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?

9.3 Dressed photon states (Exam 2011)

A photon is interacting with an atom within a small reflecting cavity. The photon energy is close to the excitation energy of the atom, which connects the ground state to the first excited state. Due to interaction between the photon and the atom the stationary states of the composite system are admixtures of the photon and atomic states. These are sometimes referred to as a ”dressed” photon states. In this problem we examine some of the properties of the dressed states.

The Hamiltonian of the photon-atom system can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \tag{1}$$

where $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The Pauli matrices act between the two atomic levels, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. $\hat{a}$ and $\hat{a}^\dagger$ are the photon creation and destruction operators.

a) We introduce the notation $|+,0\rangle = |+\rangle \otimes |0\rangle$ and $|-,1\rangle = |-\rangle \otimes |1\rangle$ for the relevant product states of the composite system, with $0, 1$ referring to the photon number. Show that in the two-dimensional subspace spanned by these vectors the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1} \tag{2}$$

where we assume $|-,1\rangle$ to correspond to the lower matrix position and $|+,0\rangle$ to the upper one. $\mathbb{1}$ denotes the 2×2 identity matrix. Express the parameters Δ, $\cos\phi$, $\sin\phi$, and ϵ in terms of ω_0, ω and λ.

b) Find the energy eigenvalues $E_\pm$. Find also the eigenstates $|\psi_\pm(\phi)\rangle$, expressed in terms of the product states $|+,0\rangle$ and $|-,1\rangle$, and show that they are related by $|\psi_-(\phi)\rangle = |\psi_+(\phi+\pi)\rangle$.

In the following we focus on the state $|\psi_-(\phi)\rangle$, which we assume to be the one-photon state in the non-interacting case. This state (with a convenient choice of phase factor) can be written as $|\psi_-(\phi)\rangle = \cos\frac{\phi}{2}\,|-,1\rangle + i\sin\frac{\phi}{2}\,|+,0\rangle$.

c) Find expressions for the reduced density operators of the photon and of the atom for the state $|\psi_-(\phi)\rangle$. Discuss in what parameter interval the state is mostly a photon-like state and when it is mostly an atom-like state.

d) Determine the entanglement entropy as a function of ϕ, and find for what values the entropy is minimal and maximal. Relate this to the discussion in c).

e) At time $t = 0$ a single photon is sent into the cavity, where there is previously no photon and the atom is in its ground state. Determine the time dependent probability $p(t)$ for a photon later to be present in the cavity. Give a qualitative explanation of its oscillatory behaviour, and specify what determines the frequency and amplitude of the oscillations.