Solutions to problem set 9

9.1 Quantum gates for teleportation

a) We have to calculate the action of each gate on the state. The initial state is

$$|\psi_0\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes |0\rangle \otimes |0\rangle = c_0|000\rangle + c_1|100\rangle.$$

We write H^i for the Hadamard gate on qubit i, and C_{NOT}^{ij} for the CNOT gate with i as control bit and j as target bit. After each gate we then get

$$|\psi_1\rangle = H^b|\psi_0\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|010\rangle + c_1|100\rangle + c_1|110\rangle\Big]$$

$$|\psi_2\rangle = C_{NOT}^{bc}|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|011\rangle + c_1|100\rangle + c_1|111\rangle\Big]$$

$$|\psi_3\rangle = C_{NOT}^{ab}|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|011\rangle + c_1|110\rangle + c_1|101\rangle\Big]$$

$$\begin{aligned}|\psi_4\rangle = H^a|\psi_3\rangle = \frac{1}{2}\Big[&c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle \\ &+c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle\Big]\end{aligned}$$

$$\begin{aligned}|\psi_5\rangle = C_{NOT}^{bc}|\psi_4\rangle = \frac{1}{2}\Big[&c_0|000\rangle + c_0|100\rangle + c_0|010\rangle + c_0|110\rangle \\ &+c_1|011\rangle - c_1|111\rangle + c_1|001\rangle - c_1|101\rangle\Big]\end{aligned}$$

$$\begin{aligned}|\psi_6\rangle = H^c|\psi_5\rangle = \frac{1}{2\sqrt{2}}\Big[&(c_0+c_1)|000\rangle + (c_0-c_1)|001\rangle + (c_0-c_1)|100\rangle + (c_0+c_1)|101\rangle \\ &+(c_0+c_1)|010\rangle + (c_0-c_1)|011\rangle + (c_0-c_1)|110\rangle + (c_0+c_1)|111\rangle\Big]\end{aligned}$$

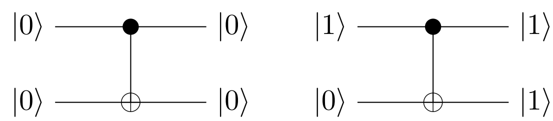
$$\begin{aligned}|\psi_7\rangle = C_{NOT}^{ac}|\psi_6\rangle = \frac{1}{2\sqrt{2}}\Big[&(c_0+c_1)|000\rangle + (c_0-c_1)|001\rangle + (c_0-c_1)|101\rangle + (c_0+c_1)|100\rangle \\ &+(c_0+c_1)|010\rangle + (c_0-c_1)|011\rangle + (c_0-c_1)|111\rangle + (c_0+c_1)|110\rangle\Big]\end{aligned}$$

$$\begin{aligned}|\psi_8\rangle = H^c|\psi_7\rangle &= \frac{1}{2}\Big[c_0|000\rangle + c_1|001\rangle + c_0|100\rangle + c_1|101\rangle \\ &\quad+c_0|010\rangle + c_1|011\rangle + c_0|110\rangle + c_1|111\rangle\Big] \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (c_0|0\rangle + c_1|1\rangle)\end{aligned}$$

b) Measuring qubits a and b at the dashed line collapses the wavefunction at that point. But since a and b only acts as control bits forthe last four gates, their states do not change. Then the state will be the same as if we measure a and b on the final state $|\psi_8\rangle$ instead. The only difference is that now the CNOT gates will not be nonlocal two-qubit gates, but rather local one-qubit gates on qubit c conditioned on the measurement outcomes for a and b. This has to be transmitted from a and b to c as in the usual teleportation protocol. Then we still get $|c'\rangle = |a\rangle$ at the end. and only need local operations after the dashed line.

9.2 Quantum cloning of orthogonal states

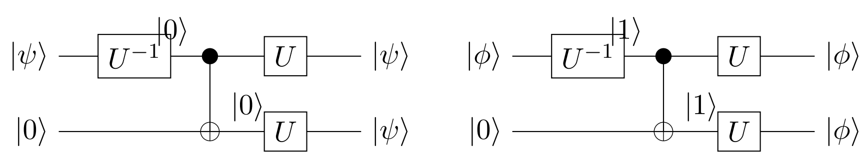
a) Assume first that $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |1\rangle$. Then we can check that a single CNOT gate gives the desired result (upper line is the original, lower line is the copy)



Since $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, there exist a unitary transformation U such that

$$|\psi\rangle = U|0\rangle$$
$$|\phi\rangle = U|1\rangle$$

The inverse of this transforms $|\psi\rangle$ and $|\phi\rangle$ to $|0\rangle$ and $|1\rangle$, and we can then use the CNOT as above and transform the result back, giving the final circuit



b) Here we can use the simple circuit with a single CNOT gate. The input is (qubits are written from top down)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

giving the final state

$$|\psi_1\rangle = C_{NOT}|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

9.3 Dressed photon states (Exam 2011)

Information:

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda\left(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+\right)$$

$$\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right), \quad \sigma_z|\pm\rangle = \pm|\pm\rangle, \quad |+,0\rangle = |+\rangle\otimes|0\rangle, \quad |-,1\rangle = |-\rangle\otimes|1\rangle$$

where $\sigma_\pm$ acts on the particle, and $\hat{a}^\dagger$,$\hat{a}$ acts on the photon state.

a)

$$H = \begin{pmatrix} \langle +,0|\hat{H}|+,0\rangle & \langle +,0|\hat{H}|-,1\rangle \\ \langle -,1|\hat{H}|+,0\rangle & \langle -,1|\hat{H}|-,1\rangle \end{pmatrix}$$

$$\begin{aligned}
\langle +,0|\hat{H}|+,0\rangle &= \langle +,0|\frac{1}{2}\hbar\omega_0\sigma_z|+,0\rangle = \frac{1}{2}\hbar\omega_0 \\
\langle +,0|\hat{H}|-,1\rangle &= \langle +,0| - i\hbar\lambda\hat{a}\sigma_+|-,1\rangle = -i\hbar\lambda \\
\langle -,1|\hat{H}|+,0\rangle &= \langle -,1|i\hbar\lambda\hat{a}^\dagger\sigma_-|+,0\rangle = i\hbar\lambda \\
\langle -,1|\hat{H}|-,1\rangle &= \langle -,1|\left(\frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a}\right)|-,1\rangle = -\frac{1}{2}\hbar\omega_0 + \hbar\omega
\end{aligned}$$

$$H = \frac{1}{2}\hbar\begin{pmatrix} \omega_0 & -2i\lambda \\ 2i\lambda & 2\omega - \omega_0 \end{pmatrix} = \frac{1}{2}\hbar\Delta\begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1}$$

with

$$\epsilon = \frac{1}{2}\hbar\omega, \quad \cos\phi = \frac{\omega_0 - \omega}{\Delta}, \quad \sin\phi = \frac{2\lambda}{\Delta}, \quad \Delta = \sqrt{(\omega_0-\omega)^2 + 4\lambda^2}$$

b) We observe that the Hamiltonian is of the form

$$H = \frac{1}{2}\hbar\Delta\sigma\cdot\mathbf{n} + \epsilon\mathbb{1}$$

with $\mathbf{n} = (0, \sin\phi, \cos\phi)$. We know that the eigenvalues of $\sigma\cdot\mathbf{n}$ are ±1, and therefore we get

$$E_\pm = \epsilon\pm = \frac{1}{2}\hbar\Delta$$

with the corresponding eigenvectors

$$|\psi_+(\phi)\rangle = \begin{pmatrix} i\cos\frac{\phi}{2} \\ -\sin\frac{\phi}{2} \end{pmatrix} \qquad |\psi_-(\phi)\rangle = \begin{pmatrix} i\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix}$$

from here, we get using:

$$\begin{aligned}
\sin\left(\frac{\phi}{2} + \frac{\pi}{2}\right) &= \cos\frac{\phi}{2} \\
\cos\left(\frac{\phi}{2} + \frac{\pi}{2}\right) &= = -\sin\frac{\phi}{2}
\end{aligned}$$

That

$$|\psi_+(\phi+\pi)\rangle = \begin{pmatrix} i\cos\frac{\phi+\pi}{2} \\ -\sin\frac{\phi+\pi}{2} \end{pmatrix} = \begin{pmatrix} -i\sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} \end{pmatrix} = -|\psi_-(\phi)\rangle$$

with the extra $-$ a phase that comes from the choice of the phase on the eigenstates which is always a free choice.

c) The density operator for $|\psi_-(\phi)\rangle$ is:

$$\begin{aligned}
\rho(\phi) &= |\psi_-(\phi)\rangle\langle\psi_-(\phi)| \\
&= \left(i\sin\frac{\phi}{2}|0,+\rangle + \cos\frac{\phi}{2}|1,-\rangle\right)\left(-i\sin\frac{\phi}{2}\langle 0,+| + \cos\frac{\phi}{2}\langle 1,-|\right) \\
&= \sin^2\frac{\phi}{2}|0,+\rangle\langle 0,+| + \cos^2\frac{\phi}{2}|1,-\rangle\langle 1,-| + i\sin\frac{\phi}{2}\cos\frac{\phi}{2}\left(|0,+\rangle\langle 1,-| - |1,-\rangle\langle 0,+|\right)
\end{aligned}$$

The partial traces are the sum of the diagonal terms in each subsystem:

$$\begin{aligned}
\rho_{ph}(\phi) &= \sin^2\frac{\phi}{2}|0\rangle\langle 0| + \cos^2\frac{\phi}{2}|1\rangle\langle 1| \\
\rho_{atom}(\phi) &= \sin^2\frac{\phi}{2}|+\rangle\langle +| + \cos^2\frac{\phi}{2}|-\rangle\langle -|
\end{aligned}$$

The photon state is given by $|-,1\rangle$ and the excited atomic state is given by $|+,0\rangle$. So when $\sin^2\frac{\phi}{2} > \cos^2\frac{\phi}{2}$, or $\phi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the state is mostly an excited atom state. When $\sin^2\frac{\phi}{2} < \cos^2\frac{\phi}{2}$, or $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the state is mostly a photon state.

d) Both ρ_{ph} and ρ_{atom} have the same eigenvalues (denoted λ below), and their von Neumann entropy is the same, the entanglement is thus given by:

$$\begin{aligned}
S &= -\operatorname{Tr}\rho_{ph}\log\rho_{ph} = -\sum_i \lambda_i\log\lambda_i \\
&= -\sin^2\frac{\phi}{2}\log\left(\sin^2\frac{\phi}{2}\right) - \cos^2\frac{\phi}{2}\log\left(\cos^2\frac{\phi}{2}\right)
\end{aligned}$$

The minimal value happens when:

$$\sin^2\frac{\phi}{2}\log\left(\sin^2\frac{\phi}{2}\right) = -\cos^2\frac{\phi}{2}\log\left(\cos^2\frac{\phi}{2}\right)$$

By inspection, we see this happens when $\phi = 0$ or $\phi = \pi$. This corresponds to

$$\rho(0) = |1,-\rangle\langle 1,-|, \quad \rho(\pi) = |0,+\rangle\langle 0,+|$$

which are separable states ($S = 0$). The maximal value on the other hand happens when:

$$\sin^2\frac{\phi}{2}\log\left(\sin^2\frac{\phi}{2}\right) = \cos^2\frac{\phi}{2}\log\left(\cos^2\frac{\phi}{2}\right)$$

This occurs when $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$. This refers to

$$\begin{aligned}
\rho\left(\frac{\pi}{2}\right) &= \frac{1}{2}|0,+\rangle\langle 0,+| + \frac{1}{2}|1,-\rangle\langle 1,-| + \frac{1}{2}i\left(|0,+\rangle\langle 1,-| - |1,-\rangle\langle 0,+|\right) \\
\rho\left(\frac{3\pi}{2}\right) &= \frac{1}{2}|0,+\rangle\langle 0,+| + \frac{1}{2}|1,-\rangle\langle 1,-| - i\frac{1}{2}\left(|0,+\rangle\langle 1,-| - |1,-\rangle\langle 0,+|\right)
\end{aligned}$$

This is not a product state, and the reduces densities are $\rho_{ph} = \rho_{atom} = \frac{1}{2}\mathbb{1}$ which is maximally entangled $S = \log 2$. We relate this to the previous discussion by noting that the maksimal entropy happens when the system changes from being mostly an excited atomic state to being mostly a photon state, and vice versa. The minimal values are in the middle of each of the two "modes" of the system. This is consistent with what we would expect from such a system.

e) We're given:

$$|\psi(\phi,0)\rangle = |-,1\rangle$$

where the time evolved state is $|\psi(\phi,t)\rangle$. In order to obtain the time evolution off the system, we need to express this in the energy eigenbasis. In terms of these states (exercise b), we can write:

$$\begin{aligned}
|\psi(\phi,0)\rangle &= \alpha|\psi_+(\phi)\rangle + \beta|\psi_-(\phi)\rangle \\
&= \alpha\left(i\cos\frac{\phi}{2}|0,+\rangle - \sin\frac{\phi}{2}|1,-\rangle\right) + \beta\left(i\sin\frac{\phi}{2}|0,+\rangle + \cos\frac{\phi}{2}|1,-\rangle\right) \\
&= \left(\alpha i\cos\frac{\phi}{2} + \beta i\sin\frac{\phi}{2}\right)|0,+\rangle + \left(\beta\cos\frac{\phi}{2} - \alpha\sin\frac{\phi}{2}\right)|-,1\rangle
\end{aligned}$$

One can solve this by inspection, but let's do it explicitly:

$$\begin{aligned}
\alpha i\cos\frac{\phi}{2} + \beta i\sin\frac{\phi}{2} &= 0 \\
\beta\cos\frac{\phi}{2} - \alpha\sin\frac{\phi}{2} &= 1
\end{aligned}$$

$$\alpha = -\beta\tan\frac{\phi}{2} \Rightarrow \beta\cos\frac{\phi}{2} + \beta\tan\frac{\phi}{2}\sin\frac{\phi}{2} = 1$$

$$\beta = \frac{1}{\cos\frac{\phi}{2} + \tan\frac{\phi}{2}\sin\frac{\phi}{2}} = \frac{\cos\frac{\phi}{2}}{\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}} = \cos\frac{\phi}{2} \Longrightarrow \alpha = -\sin\frac{\phi}{2}$$

This gives:

$$|\psi(\phi,0)\rangle = -\sin\frac{\phi}{2}|\psi_+(\phi)\rangle + \cos\frac{\phi}{2}|\psi_-(\phi)\rangle$$

Then applying the time evolution operator:

$$\begin{aligned}
|\psi(\phi,t)\rangle &= -\sin\frac{\phi}{2}e^{-iHt/\hbar}|\psi_+(\phi)\rangle + \cos\frac{\phi}{2}e^{-iHt/\hbar}|\psi_-(\phi)\rangle \\
&= -\sin\frac{\phi}{2}e^{-iE_+t/\hbar}|\psi_+(\phi)\rangle + \cos\frac{\phi}{2}e^{-iE_-t/\hbar}|\psi_-(\phi)\rangle \\
&= -\sin\frac{\phi}{2}e^{-iE_+t/\hbar}\left(i\cos\frac{\phi}{2}|0,+\rangle - \sin\frac{\phi}{2}|1,-\rangle\right) + \cos\frac{\phi}{2}e^{-iE_-t/\hbar}\left(i\sin\frac{\phi}{2}|0,+\rangle + \cos\frac{\phi}{2}|1,-\rangle\right) \\
&= \left(-i\cos\frac{\phi}{2}\sin\frac{\phi}{2}e^{-iE_+t/\hbar} + i\sin\frac{\phi}{2}\cos\frac{\phi}{2}e^{-iE_-t/\hbar}\right)|0,+\rangle + \left(\sin^2\frac{\phi}{2}e^{-iE_+t/\hbar} + \cos^2\frac{\phi}{2}e^{-iE_-t/\hbar}\right)|1 \\
&= i\sin\frac{\phi}{2}\cos\frac{\phi}{2}\left(e^{-iE_-t/\hbar} - e^{-iE_+t/\hbar}\right)|0,+\rangle + \left(\sin^2\frac{\phi}{2}e^{-iE_+t/\hbar} + \cos^2\frac{\phi}{2}e^{-iE_-t/\hbar}\right)|1,-\rangle
\end{aligned}$$

The probability is then given by

$$
\begin{aligned}
p_{ph}(t) = \quad &= \quad |\,\langle -,1|\psi(\phi,t)\rangle\,|^2 \\
&= \quad |\,\sin^2\frac{\phi}{2}e^{-i(E_+-E_-)t/\hbar} + \cos^2\frac{\phi}{2}^2\,| \\
&= \quad \underbrace{\sin^4\frac{\phi}{2}}_{=\left[\frac{1}{2}(1-\cos\phi)\right]^2} + \underbrace{\cos^4\frac{\phi}{2}}_{=\left[\frac{1}{2}(1+\cos\phi)\right]^2} + \underbrace{2\sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2}}_{=\frac{1}{4}\sin^2\phi}\cos(\Delta t) \\
&= \quad \frac{1}{2}\left(1+\cos^2\phi+\sin^2\phi\cos\Delta t\right), \quad \Delta = \sqrt{(\omega_0-\omega)^2+4\lambda^2}
\end{aligned}
$$

where we used that $(E_+ - E_-)/\hbar = \Delta$. We see the probability oscillating with frequency Δ and amplitude $\frac{1}{2}\sin^2\phi = \frac{1}{2}\left(\frac{2\lambda}{\Delta}\right)^2 = 2\left(\frac{\lambda}{\Delta}\right)^2$ centered around $\frac{1}{2}+\frac{1}{2}\cos^2\phi = \frac{1}{2}+\frac{1}{2}\left(\frac{\omega_0-\omega}{\Delta}\right)^2$. This is due to the mixing of the excited atom state and photon state in the hamiltonian which makes the system oscillate between being mostly photon-state and mostly excited atom state.