UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 4110 Non-relativistic quantum mechanics
Day of exam: Thursday, December 4, 2008
Exam hours: 3 hours, beginning at 14:30
This examination paper consists of 3 pages
Permitted materials: Calculator
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before you begin.

PROBLEM 1
Spin half particle in a harmonic oscillator potential
A spin half particle is moving in a one-dimensional harmonic oscillator potential (in the x-direction) under the influence of a constant magnetic field (in the z-direction). The Hamiltonian is

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{1}{2}\hbar\omega_1\sigma_z + \lambda\hbar(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{1}$$

where the first term is the harmonic oscillator part with ω_0 as the oscillator frequency, the second term is the spin energy due to the magnetic field, with ω_1 as the spin precession frequency, and the third term is a coupling term between the spin and the position coordinate of the particle. The spin flip operators are defined as $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hat{a}$ and $\hat{a}^\dagger$ are the standard lowering and raising operators of the harmonic oscillator and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.

When $\lambda = 0$, the spin and position of the particle are uncoupled and the energy eigenstates are $|n, m\rangle$ with $n = 0, 1, 2, ...$ as the harmonic oscillator quantum number and $m = \pm 1$ as the spin quantum number, corresponding to spin up/down along the z-axis. When $\lambda \neq 0$, the unperturbed eigenstates will pairwise be coupled by the Hamiltonian, so that $|n, +1\rangle$ is coupled to $|n + 1, -1\rangle$.

a) Consider the two-dimensional subspace spanned by basis vectors$|0, +1\rangle$ and $|1, -1\rangle$. Show that in this space the Hamiltonian takes the form of a 2x2 matrix which can be written as

$$H = \hbar\Delta \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \hbar\epsilon\mathbb{1} \tag{2}$$

with $\mathbb{1}$ as the 2x2 identity matrix. Determine $\Delta\cos\theta$, $\Delta\sin\theta$ and ϵ.

b) Find the energies and eigenstates of H in the two-dimensional subspace, expressed as functions of Δ, θ and ϵ.

c) The basis vectors $|n,m\rangle$ can be regarded as tensor products of position and spin vectors, $|n,m\rangle = |n\rangle \otimes |m\rangle$. The two eigenstates found under b) will be entangled with respect to the position and spin variables. Determine the degree of entanglement as function of θ. What value for θ gives the smallest what gives the largest degree of entanglement?

PROBLEM 2
Electric dipole transition
We consider the transition in hydrogen from the excited 2p level to the ground state 1s, where a single photon is emitted. The initial atomic state (A) we assume to have $m = 0$ for the z-component of the orbital angular momentum, so that the quantum numbers of this state are $(n,l,m) = (2,1,0)$, with n as the principle quantum number and l as the orbital angular momentum quantum number. Similarly the ground state (B) has quantum numbers $(n,l,m) = (1,0,0)$. When expressed in polar coordinates the wave functions of the two states (with intrinsic spin of the electron not included) are given by

$$\begin{aligned}\psi_A(r,\phi,\theta) &= \frac{1}{\sqrt{32\pi a_0^3}} \cos\theta \, \frac{r}{a_0} \, e^{-\frac{r}{2a_0}} \\ \psi_B(r,\phi,\theta) &= \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}\end{aligned} \tag{3}$$

where a_0 is the Bohr radius.

We remind you about the form of the interaction matrix element in the dipole approximation,

$$\langle B, 1_{\mathbf{k}a}|\hat{H}_{emis}|A,0\rangle = ie\sqrt{\frac{\hbar\omega}{2V\epsilon_0}}\,\boldsymbol{\epsilon}^*_{\mathbf{k}a}\cdot\mathbf{r}_{BA} \tag{4}$$

where e is the electron charge, $\mathbf{k}$ is the wave vector of the photon, a is the polarization quantum number, ω is the photon frequency and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is a polarization vector. V is a normalization volume for the electromagnetic wave functions, ϵ_0 is the permittivity of vacuum and $\mathbf{r}_{BA}$ is the matrix element of the electron position operator between the initial and final atomic states.

a) Explain why the x- and y-components of $\mathbf{r}_{BA}$ vanish while the z-component has the form $z_{BA} = \nu a_0$, with ν as a numerical factor. Determine the value of ν. (A useful integration formula is $\int_0^\infty dx\, x^n\, e^{-x} = n!$.)

b) To first order in perturbation theory the interaction matrix element (4) determines the direction of the emitted photon, in the form of a probability distribution $p(\phi,\theta)$, where (ϕ,θ) are the polar angles of the wave vector $\mathbf{k}$. Determine $p(\phi,\theta)$ from the above expressions.

c) The life time of the 2p state is $\tau_{2p} = 1.6\cdot 10^{-9}s$ while the excited 2s state (with angular momentum $l = 0$) has a much longer life time, $\tau_{2s} = 0.12s$. Du you have a (qualitative) explanation for the large difference?

PROBLEM 3
Density operators and entanglement
Give a brief and concise discussion of the following points:

a) List the general properties of density operators (or density matrices) and specify the difference between a *pure* and a *mixed* state.

b) For a composite system consisting of two parts $\mathcal{A}$ and $\mathcal{B}$ use the density operator formulation to explain the difference between, uncorrelated states, states with classical correlations (separable states) and entangled states.

c) Assume the full system is in a pure state, described by the state vector $|\psi\rangle$. What is meant by the *Schmidt decomposition* of this state vector relative to the two subsystems $\mathcal{A}$ and $\mathcal{B}$? Use the decomposition to find expressions for the reduced density operators of the two subsystems, and show that the von Neumann entropy of the reduced density operators are equal.

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Eksamen i: FYS 4110 Ikke-relativistisk kvantemekanikk
Eksamensdag: Torsdag 3. desember, 2009
Tid for eksamen: kl. 14.30 (3 timer)
Oppgavesettet er på 3 sider
Tilatte hjelpemidler: Kalkulator
Øgrim og Lian eller Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

BOKMÅL

OPPGAVE 1
To-nivåsystemer

Vi studerer i det følgende et kvantemekanisk system som er sammensatt av to to-nivåsystemer $\mathcal{A}$ og $\mathcal{B}$. Hilbertrommet til det fulle systemet $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ er dermed firedimensjonalt. De to delsystemene er dynamisk koblet, og Hamiltonoperatoren for det fulle systemet har formen

$$\hat{H} = \frac{1}{2}\hbar\omega(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - i\hbar\lambda(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) \tag{1}$$

hvor σ_z og $\sigma_\pm$ er Pauli-matriser, med $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hbar\omega$ er (den like store) energi-splittingen i hvert av de to delsystemene, og λ er en koblingsparameter. I tensorproduktuttrykkene regner vi at første faktor virker på delsystem $\mathcal{A}$ og andre faktor på delsystem $\mathcal{B}$.

a) Vis at den tidsavhengige Schrödinger-ligningen har en løsning på formen

$$|\psi(t)\rangle = \cos(\lambda t)|+-\rangle + \sin(\lambda t)|-+\rangle \tag{2}$$

hvor $|+-\rangle = |+\rangle \otimes |-\rangle$ og $|-+\rangle = |-\rangle \otimes |+\rangle$ og hvor $\sigma_z|\pm\rangle = \pm|\pm\rangle$ for hvert av delsystemene. Hva blir uttrykket for den tilsvarende tetthetsoperator $\hat{\rho}(t)$ når den skrives på bra-ket form?

b) Den tidsavhengige tetthetsoperatoren kan også uttrykkes ved Pauli-matriser på en tilsvarende måte som i (1). Finn dette uttrykket, og finn også de reduserte tetthetsoperatorene $\hat{\rho}_A(t)$ og $\hat{\rho}_B(t)$, begge uttrykt ved Paulimatriser (og identitets-operatoren).

c) Angi det generelle uttrykket for graden av sammenfiltring i et sammensatt systemet når det befinner seg i en kvantemekanisk *ren tilstand*. I den tidsavhengige tilstanden beskrevet ovenfor, hva blir da uttrykket?

OPPGAVE 2
Koblete harmoniske oscillatorer
To harmoniske oscillatorer, kalt $\mathcal{A}$ og $\mathcal{B}$, behandles som et sammensatt kvantemekanisk system. Hamiltonoperatoren til systemet har formen

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + \mathbb{1}) + \hbar\lambda(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \tag{3}$$

med $(\hat{a}, \hat{a}^\dagger)$ som senke- og heveoperatorer for $\mathcal{A}$ og $(\hat{b}, \hat{b}^\dagger)$ som tilsvarende operatorer for $\mathcal{B}$. ω og λ er to reelle konstanter.

a) Vis at Hamiltonoperatoren kan skrives på diagonal form,

$$\hat{H} = \hbar\omega_c\, \hat{c}^\dagger\hat{c} + \hbar\omega_d\, \hat{d}^\dagger\hat{d} + \hbar\omega\mathbb{1} \tag{4}$$

hvor c og d er lineære kombinasjoner av a og b,

$$c = \mu\, a + \nu\, b\,, \quad d = -\nu\, a + \mu\, b \tag{5}$$

og hvor μ og ν er reelle konstanter som tilfredsstiller $\mu^2 + \nu^2 = 1$. (Tilsvarende uttrykk gjelder for de hermitisk konjugerte operatorene $\hat{c}^\dagger$ og $\hat{d}^\dagger$.) Bestem de nye parametrene ω_c, ω_d, μ og ν, uttrykt ved ω og λ. Sjekk at de nye operatorene $\hat{c}$ og $\hat{d}$ tilfredsstiller de samme kommutasjonsrelasjonene som $\hat{a}$ og $\hat{b}$ ved at $\left[\hat{c}, \hat{c}^\dagger\right] = \left[\hat{d}, \hat{d}^\dagger\right] = \mathbb{1}$ og $\left[\hat{c}, \hat{d}^\dagger\right] = 0$.

b) Anta at tilstanden $|\psi(0)\rangle$ til det sammensatte systemet ved $t = 0$ er en koherent tilstand for begge de nye variablene, slik at

$$\hat{c}\,|\psi(0)\rangle = z_{c0}|\psi(0)\rangle\,, \quad \hat{d}\,|\psi(0)\rangle = z_{d0}|\psi(0)\rangle \tag{6}$$

Tilstanden vil også på et senere tidspunkt være en koherent tilstand for $\hat{c}$ og $\hat{d}$, med egenverdier

$$z_c(t) = e^{-i\omega_c t} z_{c0}\,, \quad z_d(t) = e^{-i\omega_d t} z_{d0} \tag{7}$$

Vis dette for $z_c(t)$. (Uttrykket for $z_d(t)$ følger på samme måte, og trengs derfor ikke vises.)

c) Vis at tilstanden $|\psi(t)\rangle$ også er en koherent tilstand for de opprinnelige harmonisk oscillator-operatorene $\hat{a}$ og $\hat{b}$, og bestem egenverdiene $z_a(t)$ og $z_b(t)$ uttrykt ved initialverdiene z_{a0} og z_{b0}.

OPPGAVE 3
Harmonisk oscillator i varmebad
En harmonisk oscillator med vinkelfrekvens ω er i termisk likevekt med et varmebad med temperatur T. Den befinner seg da i en blandet kvantemekanisk tilstand uttrykt ved tetthetsoperatoren

$$\hat{\rho} = Ne^{-\beta\hat{H}} \tag{8}$$

med $\hat{H}$ som Hamiltonoperatoren til den harmoniske oscillatoren, N som en normeringskonstant og $\beta = 1/kT$ hvor k er Boltzmanns konstant.

a) Sjekk at $\hat{\rho}$ tilfredsstiller kravene til en tetthetsmatrise og bestem normaliseringskonstanten N.

b) Vis at forventningsverdien for energien kan skrives som

$$E = \mathrm{Tr}(\hat{H}\hat{\rho}) = \frac{1}{N}\frac{dN}{d\beta} \tag{9}$$

og finn energien som funksjon av β. Vis at for lav temperatur, $T \to 0$ eller $\beta \to \infty$, vil energien nærme seg grunntilstandsenergien til oscillatoren.

c) Tetthetsoperatoren kan skrives på diagonal form som

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle\langle n| \tag{10}$$

hvor $p_n = N e^{-\beta\hbar\omega(n+1/2)}$. Det vil si at vi kan se på tilstanden $\hat{\rho}$ som en statistisk blanding av energiegentilstander $|n\rangle$, vektet med sannsynlighetene p_n. Den samme tilstanden kan imidlertid også ses på som en statistisk blanding av koherente tilstander, på formen

$$\hat{\rho} = \int \frac{d^2 z}{\pi}\, p(|z|)\, |z\rangle\langle z| \tag{11}$$

hvor $p(|z|)$ er en sannsynlighetsfunksjon som bare avhenger av absoluttverdien $|z| = r$.

Vis at uttrykket (11) kan omformuleres til (10), og finn p_n uttrykt ved $p(r)$.

Vi minner om følgende uttrykk:

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\, e^{-\frac{1}{2}|z|^2} \tag{12}$$

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Eksamensdag: Onsdag 8. desember, 2010
Tid for eksamen: kl. 14.30 (4 timer)
Oppgavesettet er på 4 sider
Tilatte hjelpemidler: Kalkulator
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OPPGAVE 1
Sammenfiltring i et trepartikkel-system

Tre partikler med halvtallig spinn, som vi referer til som $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, befinner seg i en sammensatt spinntilstand

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle - |ddd\rangle) \tag{1}$$

hvor $|uuu\rangle = |u\rangle_{\mathcal{A}} \otimes |u\rangle_{\mathcal{B}} \otimes |u\rangle_{\mathcal{C}}$, er tensorproduktet av *spinn-opp (u)* langs z-aksen for alle tre partiklene, mens $|ddd\rangle = |d\rangle_{\mathcal{A}} \otimes |d\rangle_{\mathcal{B}} \otimes |d\rangle_{\mathcal{C}}$ er tensorprodukttilstanden svarende til *spinn-ned (d)* for alle tre partiklene. Vi antar at posisjonskoordinatene er helt frakoblet spinnkoordinatene og at spinnet derfor kan studeres separat.

a) Forklar hva vi mener med at de tre partiklene er i en *korrelert* spinntilstand, og hva vi mener med at spinnene er *sammenfiltret*.

Vi studerer i det følgende spinnsystemet som todelt, svarende til en oppsplitting $\mathcal{ABC} = \mathcal{A} + \mathcal{BC}$, slik at spinn $\mathcal{A}$ definerer det ene undersystemet og de to andre spinnene, $\mathcal{B}$ og $\mathcal{C}$, definerer det andre undersystemet.

b) Bestem de reduserte tetthetsoperatorene $\hat{\rho}_{\mathcal{A}}$ og $\hat{\rho}_{\mathcal{BC}}$ for de to delsystemene. Forklar hva vi mener med sammenfiltringsentropien til et todelt, sammensatt system og bestem verdien på denne for spinntilstanden (1). Hva menes med at spinntilstanden er en *maksimalt* sammenfiltret tilstand?

Tilstanden til delsystem $\mathcal{BC}$ er beskrevet av tetthetsoperatoren $\hat{\rho}_{\mathcal{BC}}$. Hva sier denne om sammenfiltring mellom de to spinnene $\mathcal{B}$ and $\mathcal{C}$?

Anta nå at de tre partiklene $\mathcal{A}$, $\mathcal{B}$ og $\mathcal{C}$ tas hånd om av tre fysikere (også identifisert som $\mathcal{A}$, $\mathcal{B}$ og $\mathcal{C}$) som befinner seg på forskjellige steder, men som er i stand til å beskytte tilstanden til hver sin partikkel slik at den totale spinntilstanden (1) ikke forandrer seg før en av dem foretar en spinnmåling.

c) Ved et gitt tidspunkt måler $\mathcal{A}$ spinnkomponenten langs x-aksen for sin partikkel og finner spinn-opp som måleresultatet. (Spinn-opp-tilstanden langs x-aksen blir betegnet $|f\rangle$ og spinn-ned-tilstanden $|b\rangle$.) Hun sender beskjed om dette til $\mathcal{B}$ og $\mathcal{C}$, og disse beregner, med utgangspunkt i denne opplysningen, den nye tetthetsoperatoren $\hat{\rho}$ for det fulle systemet og bestemmer den nye reduserte tetthetsoperator $\hat{\rho}_{\mathcal{BC}}$.

Hva er de nye uttrykkene for tetthetsoperatorene $\hat{\rho}$ og $\hat{\rho}_{\mathcal{BC}}$? Er det noen endring i sammenfiltringen mellom spinnene $\mathcal{B}$ og $\mathcal{C}$?

OPPGAVE 2
Spinnflipp-stråling

Vi studerer i denne oppgaven overgang mellom to spinntilstander for et elektron i et ytre magnetfelt som er rettet langs z-aksen, $\mathbf{B} = B\mathbf{e}_z$. (Merk: vi benytter her $\mathbf{e}_x$, $\mathbf{e}_y$ og $\mathbf{e}_z$ som enhetsvektorer langs x-, y- og z-aksen, siden $\mathbf{k}$ benyttes som bølgevektoren for det utsendte fotonet.) Hamiltonoperatoren skrives som $\hat{H} = \hat{H}_0 + \hat{H}_1$, hvor $\hat{H}_0$ svarer til den magnetiske dipolenergien i det ytre magnetfeltet, mens $\hat{H}_1$ beskriver koblingen mellom spinnet og strålingsfeltet. Vi har

$$\hat{H}_0 = \frac{1}{2}\omega_B \sigma_z\,, \qquad \omega_B = -\frac{eB}{m} \tag{2}$$

med e som elektronladningen og m som elektronmassen. Frekvensen ω_B regnes som positiv.

Matriseelementet til spinnvekselvirkningen $\hat{H}_1$ ved emisjon av ett foton er i dipoltilnærmelsen

$$\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle = i\frac{e\hbar}{2m}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\,(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}a})\cdot\boldsymbol{\sigma}_{BA} \tag{3}$$

hvor $|A\rangle$ er den eksiterte spinntilstanden (spinn-opp) og $|B\rangle$ er grunntilstanden (spinn-ned). Videre er $\boldsymbol{\epsilon}_{\mathbf{k}a}$ en polarisasjonsvektor og $\omega = c\,k$ er sirkelfrekvensen til det emitterte fotonet, V er et normeringsvolum for den elektromagnetiske strålingen og $\boldsymbol{\sigma}_{AB}$ er matriseelementet til Paulimatrisen $\boldsymbol{\sigma} = \sigma_x\,\mathbf{e}_x + \sigma_y\,\mathbf{e}_y + \sigma_z\,\mathbf{e}_z$ mellom de to spinntilstandene. Vi minner om formen på Paulimatrisene,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

a) Til første orden i perturbasjonsteori vil vinkelavhengigheten til det kvadrerte matriseelementet $|\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle|^2$ bestemme sannsynlighetsfordelingen for retningen til fotonet, $p(\phi,\theta)$, hvor (ϕ,θ) er polarvinklene til bølgevektoren $\mathbf{k}$. Bestem $p(\phi,\theta)$ fra uttrykket ovenfor. Vi minner om at ved summasjon over polarisasjonsretningene har vi $\sum_a |\boldsymbol{\epsilon}_{\mathbf{k}a}\cdot\mathbf{b}|^2 = |\mathbf{b}|^2 - |\mathbf{b}\cdot\frac{\mathbf{k}}{k}|^2$ for en vilkårlig vektor $\mathbf{b}$. Normeringen av sannsynlighetsfordelingen er $\int d\phi \int d\theta \sin\theta\, p(\phi,\theta) = 1$.

b) Det kvadrerte matriseelementet bestemmer også, for gitt $\mathbf{k}$, sannsynlighetsfordelingen over polarisasjonsretningen til fotonet. Anta at en fotondetektor registrerer fotoner utsendt med retning langs x-aksen ($\mathbf{k} = k\mathbf{e}_x$) og med polarisasjon langs polarisasjonsvektoren $\boldsymbol{\epsilon}(\alpha) = \cos\alpha\,\mathbf{e}_y + \sin\alpha\,\mathbf{e}_x$. Hva er sannsynlighetsfordelingen $p(\alpha)$ for å detektere det emitterte fotonet, som funksjon av vinkelen α? (Anta også her at fordelingen er normert til 1, dvs. den beskriver sannsynlighet for forskjellige polarisasjonstilstander, forutsatt at fotonet er emitert langs x-aksen.)

c) Til en god tilnærmelse vil besetningssannsynligheten for den eksiterte spinntilstanden reduseres eksponensielt med tiden

$$P_A(t) = e^{-t/\tau_A} \tag{5}$$

hvor levetiden τ_A til første orden i vekselvirkningen er bestemt av (den tidsuavhengige) overgangsraten

$$w_{BA} = \frac{V}{(2\pi\hbar)^2} \int d^3k \sum_a |\langle B, 1_{\mathbf{k}a} | \hat{H}_1 | A, 0\rangle|^2 \delta(\omega - \omega_B) \tag{6}$$

Benytt dette til å finne et uttrykk for levetiden τ_A.

OPPGAVE 3
En tvungen harmonisk oscillator

En kvantemekanisk, tvungen harmonisk oscillator er beskrevet ved en Hamiltonoperator på formen

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \tag{7}$$

hvor $\hat{a}$ og $\hat{a}^\dagger$ oppfyller standard kommutasjonsrelasjoner for heve og senke-operatorer, og hvor ω_0, ω og λ er tre konstanter. Vi innfører dimensjonsløse posisjons og bevegelsemengde-operatorer som

$$\hat{x} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)\,, \quad \hat{p} = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger) \tag{8}$$

a) Vi minner om den generelle form på Heisenbergs bevegelsesligning,

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar}\left[H, \hat{A}\right] + \frac{\partial}{\partial t}\hat{A} \tag{9}$$

for en observabel $\hat{A}$. Benytt denne på senkeoperatoren $\hat{a}$, utled en bevegelsesligning for $\hat{x}$ på formen

$$\frac{d^2\hat{x}}{dt^2} + \omega_0^2\hat{x} = C\cos\omega t \tag{10}$$

og bestem konstanten C.

b) Ved å anvende følgende tidsavhengige, unitære transformasjon

$$\hat{T}(t) = e^{i\omega t\,\hat{a}^\dagger\hat{a}} \tag{11}$$

vil den nye Hamiltonoperatoren, $\hat{H}_T(t)$, som bestemmer tidsutviklingen til de transformerte tilstandsvektorene $|\psi_T(t)\rangle = \hat{T}(t)|\psi(t)\rangle$, bli tidsuavhengig. Finn utrykket for denne operatoren.

c) En koherent tilstand er definert som en egentilstand for senkeoperatoren $\hat{a}$,

$$\hat{a}|z\rangle = z|z\rangle \tag{12}$$

Anta at ved tiden $t = 0$ er oscillatoren i grunntilstanden for den λ-uavhengige del av Hamilton-operatoren, dvs.

$$|\psi(0)\rangle = |0\rangle\,, \qquad \hat{a}|0\rangle = 0 \tag{13}$$

Vis at den fortsetter å være i en koherent tilstand under tidsutviklingen altså slik at

$$|\psi(t)\rangle = e^{i\alpha(t)}|z(t)\rangle \tag{14}$$

med $\alpha(t)$ er en tidsavhengig fase og $z(t)$ som en kompleks tidsavhengig funksjon.

Bestem funksjonen $z(t)$ og gi en kvalitativ beskrivelse av bevegelsen i det komplekse z-planet. Vis at realdelen $x(t) = (z(t) + z(t)^*)/2$ oppfyller samme bevegelsesligning (10) som posisjonsoperatoren $\hat{x}(t)$.

Vi minner om operatorrelasjonen

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + ... \tag{15}$$

som gjelder for to vilkårlig valgte operatorer $\hat{A}$ og$\hat{B}$.

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 4110/ 9110 Non-relativistic quantum mechanics
Day of exam: Wednesday, December 7, 2011
Exam hours: 4 hours, beginning at 14:30
This examination paper consists of 2 problems on 3 pages
Permitted materials: Calculator
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
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PROBLEM 1
Dressed photon states

A photon is interacting with an atom within a small reflecting cavity. The photon energy is close to the excitation energy of the atom, which connects the ground state to the first excited state. Due to interaction between the photon and the atom the stationary states of the composite system are admixtures of the photon and atomic states. These are sometimes referred to as a ”dressed” photon states. In this problem we examine some of the properties of the dressed states.

The Hamiltonian of the photon-atom system can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \tag{1}$$

where $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The Pauli matrices act between the two atomic levels, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. $\hat{a}$ and $\hat{a}^\dagger$ are the photon creation and destruction operators.

a) We introduce the notation $|+,0\rangle = |+\rangle \otimes |0\rangle$ and $|-,1\rangle = |-\rangle \otimes |1\rangle$ for the relevant product states of the composite system, with $0, 1$ referring to the photon number. Show that in the two-dimensional subspace spanned by these vectors the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1} \tag{2}$$

where we assume $|-,1\rangle$ to correspond to the lower matrix position and $|+,0\rangle$ to the upper one. $\mathbb{1}$ denotes the 2×2 identity matrix. Express the parameters Δ, $\cos\phi$, $\sin\phi$, and ϵ in terms of ω_0, ω and λ.

b) Find the energy eigenvalues $E_\pm$. Find also the eigenstates $|\psi_\pm(\phi)\rangle$, expressed in terms of the product states $|+,0\rangle$ and $|-,1\rangle$, and show that they are related by $|\psi_-(\phi)\rangle = |\psi_+(\phi+\pi)\rangle$.

In the following we focus on the state $|\psi_-(\phi)\rangle$, which we assume to be the one-photon state in the non-interacting case. This state (with a convenient choice of phase factor) can be written as $|\psi_-(\phi)\rangle = \cos\frac{\phi}{2}\,|-,1\rangle + i\sin\frac{\phi}{2}\,|+,0\rangle$.

c) Find expressions for the reduced density operators of the photon and of the atom for the state $|\psi_-(\phi)\rangle$. Discuss in what parameter interval the state is mostly a photon-like state and when it is mostly an atom-like state.

d) Determine the entanglement entropy as a function of ϕ, and find for what values the entropy is minimal and maximal. Relate this to the discussion in c).

e) At time $t = 0$ a single photon is sent into the cavity, where there is previously no photon and the atom is in its ground state. Determine the time dependent probability $p(t)$ for a photon later to be present in the cavity. Give a qualitative explanation of its oscillatory behaviour, and specify what determines the frequency and amplitude of the oscillations.

PROBLEM 2
A radiation problem

We consider a one-dimensional problem where a two-level system (A) interacts with a scalar radiation field (B). The notation we use is essentially the same as in Problem 1. The Hamiltonian of the system we consider is

$$\hat{H} = \frac{1}{2}\hbar\omega_A\,\sigma_z + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \kappa \sum_k \sqrt{\frac{\hbar}{2L\omega_k}}(\hat{a}_k\sigma_+ + \hat{a}_k^\dagger\sigma_-) = \hat{H}_0 + \hat{H}_{int} \tag{3}$$

The first term is the two-level Hamiltonian, with energy splitting $\hbar\omega_A$, the second one is the free field contribution, with $k = 2\pi n/L$ (n - integer) as the wave number of the photon. L is a (large) normalization length. The third term is the interaction term $\hat{H}_{int}$, with κ as an interaction parameter. The frequency parameter is $\omega_k = ck$.

a) A general state of the two-level system is characterized by a vector $\mathbf{r}$, with $r \leq 1$, and with the corresponding density matrix as

$$\rho_A = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{4}$$

Consider first that the interaction term $\hat{H}_{int}$ is turned off, $\kappa = 0$, so that the time evolution operator of the two-level system is $\hat{\mathcal{U}}(t) = \exp(-\frac{i}{2}\omega_A\,t\,\sigma_z)$. Use this to determine the the density matrix $\rho_A(t)$ at time t, assuming that $\rho_A(0)$ is identical to the density matrix in (4), and show that the time evolution of $\mathbf{r}$ is a precession around the z-axis with angular velocity ω_A.

b) Assume next that $\kappa \neq 0$ and that initially the two-level system is in the excited "spin up state", while the scalar field is in the vacuum state. Thus, the initial state is $|+,0\rangle = |+\rangle \otimes |0\rangle$. It decays to the "spin down state" by emission of a field quantum. The final state we then write as $|-,1_k\rangle = |-\rangle \otimes |1_k\rangle$.

The occupation probability of the excited state $|+\rangle$ decays exponentially, $P_+(t) = \exp(-\gamma t)$, with a decay rate γ that to first order in the interaction, and in the limit $L \to \infty$, is given by

$$\gamma = \frac{L}{(2\pi\hbar)^2}\int dk\,|\langle -,1_k|\hat{H}_{int}|+,0\rangle|^2\,\delta(\omega_k - \omega_A) \tag{5}$$

Determine the decay rate γ, expressed in terms of the parameters of the problem.

As discussed in the lectures an approximate way to handle the decay is to introduce an imaginary contribution to the energy of the decaying state. Assuming a more general initial state, of the form

$$|\psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle) \otimes |0\rangle = \alpha|+,0\rangle + \beta|-,0\rangle \tag{6}$$

with α and β as unspecified coefficients, with $|\alpha|^2 + |\beta|^2 = 1$, we make the corresponding *ansatz* for the time evolved state

$$|\psi(t)\rangle = (e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle) \otimes |0\rangle + \sum_k c_k(t)|-,1_k\rangle \tag{7}$$

with $c_k(t)$ as decay parameters, which satify $c_k(0) = 0$.

c) Check what normalization of the state vector (7) means for the decay parameters, and determine the reduced density matrix matrix $\rho_A(t)$ of the two-level system.

d) Assume the same initial conditions as in b), $z(0) = 1$, $x(0) = y(0) = 0$ ($\alpha = 1$, $\beta = 0$). Determine the density matrix $\rho_A(t)$ and the corresponding time dependent vector $\mathbf{r}(t)$. Is the time evolution consistent with the expected exponential decay of the excited state of the two-level system? Give a brief description of the evolution of the entanglement between the two level system and the radiation field during the decay.

e) Choose another initial condition $x(0) = 1$, $y(0) = z(0) = 0$ ($\alpha = \beta = 1/\sqrt{2}$), and find also in this case the time evolution of the reduced density matrix and the components of the vector $\mathbf{r}(t)$. Sketch the time evolution of $\mathbf{r}(t)$ and compare qualitatively the motion with that in a) and d). Find $r(t)^2$ expressed as a function of γt, and sketch also this function. What does it show about the time evolution of the entanglement between the two subsystems A and B?

Assume in this paragraph $\gamma << \omega_A$.

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 4110 Non-relativistic quantum mechanics
Day of exam: Friday, December 7, 2012
Exam hours: 4 hours, beginning at 14:30
This examination paper consists of 3 problems on 4 pages
Permitted materials: Calculator
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before you begin.

PROBLEM 1
Two spin-half systems

A quantum system is composed of two interacting spin-half systems. The Hamiltonian has the form

$$\hat{H} = \frac{1}{2}\hbar\omega_1\, \sigma_z \otimes \mathbb{1} + \frac{1}{2}\hbar\omega_2\, \mathbb{1} \otimes \sigma_z + \frac{1}{2}\hbar\lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

where σ_z og $\sigma_\pm$ are Pauli matrices, with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hbar\omega_1$ and $\hbar\omega_2$ giving the splitting between the two energy levels of each of the spins, and with λ as a coupling parameter. The two factors of the tensor product refer to each of the two spin systems. We define the frequency difference as $\Delta = \omega_1 - \omega_2$ and introduce the following parametrization, $\Delta = \mu\cos\phi$ and $\lambda = \mu\sin\phi$. We further use $|\pm\rangle$ as notation for the eigenstates of σ_z. In the following we use the tensor products of these states as basis for the Hilbert space of the composite system.

a) Show that only the product states $|+-\rangle = |+\rangle \otimes |-\rangle$ and $|-+\rangle = |+\rangle \otimes |-\rangle$ are mixed by the λ term in the Hamiltonian, and show that the mixing coefficients only depend on the angle ϕ, which we will assume to lie in the interval $0 \leq \phi \leq \pi/2$. Give the expression for the Hamiltonian as a 2x2 matrix, when restricted to the subspace spanned by $|+-\rangle$ and $|-+\rangle$.

b) Find the corresponding two energy eigenvalues, and find the eigenstates expressed as functions of ϕ.

c) We now assume $\Delta = 0$. At time $t = 0$ the system is in the state $|+-\rangle$. Determine the time evolution of the state vector and the corresponding reduced density matrices for the two subsystems. Show that the entanglement entropy has a periodic behavior. What are the maximum and minimum values and what is the period of the oscillations.

PROBLEM 2
Atom-photon interaction in a cavity

An atom is trapped inside a small reflecting cavity. The energy difference between the ground state and the first excited state is $\Delta E = E_e - E_g \equiv \hbar\omega$, with ω matching the frequency of one of the electromagnetic cavity modes. This gives a strong coupling between the atomic states and this cavity mode, while the couplings to the other cavity modes are weak and can be neglected.

The composite system, atom plus cavity mode, is described by the following effective Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) - i\gamma\hbar a^\dagger a \tag{2}$$

where the Pauli matrices act between the two atomic levels, with σ_z being diagonal in the energy basis, and $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ being matrices that raise or lower the atomic energy. $\hat{a}^\dagger$ and $\hat{a}$ are the photon creation and destruction operators. λ is an interaction parameter and γ is a decay parameter. The decay is due to the process where the photon escapes through the cavity walls. Both λ and γ are real-valued parameters, and we assume $\gamma << \lambda$ and $\gamma << \omega$.

We characterize the relevant states of the composite system as $|g, 0\rangle$, $|g, 1\rangle$ and $|e, 0\rangle$, where g refers to the atomic ground state, e to the excited state, and 0 and 1 refers to the absence or presence of a photon in the cavity mode.

a) Show that in the two-dimensional subspace spanned by the vectors $|g, 1\rangle$ and $|e, 0\rangle$ the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} -i\gamma & \lambda \\ \lambda & i\gamma \end{pmatrix} \tag{3}$$

where $|g, 1\rangle$ corresponds to the upper row of the matrix and $|e, 0\rangle$ to the lower one, and $\mathbb{1}$ is the identity matrix.

b) Assume that initially the system is in the state $|\psi(0)\rangle = |e, 0\rangle$. Show that the time evolution of the state vector can be written as

$$|\psi(t)\rangle = e^{-\frac{i}{2}\omega t - \frac{1}{2}\gamma t}\left((\cos(\Omega t) + a\sin(\Omega t))|e, 0\rangle + ib\sin(\Omega t)|g, 1\rangle\right) \tag{4}$$

and determine the constants Ω, a and b.

c) Denote the corresponding density operator as $\hat{\rho}(t)$. The norm of this operator is not conserved, but if we add a contribution

$$\hat{\rho}_{tot}(t) = \hat{\rho}(t) + f(t)|g, 0\rangle\langle g, 0| \tag{5}$$

then the norm is conserved, with value 1, for a particular function $f(t)$. Determine this function, and comment on in what sense the addition of the last term in (5) is reasonable, when considering the physical process described by the Hamiltonian (3). Give a short qualitative description of the process described by (5).

PROBLEM 3
Distributed information

A secret message is distributed to a party of three, denoted A, B, and C, in the form of an entangled three-spin state, coded into three spin-half particles. As the receiving party knows in advance, the quantum state is one out of a selection of three,

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}(|+--\rangle + \eta^n|-+-\rangle + (\eta^*)^n|--+\rangle)\,, \quad \eta = e^{2\pi i/3} \tag{6}$$

where $n = 0, 1, 2$. The message is identified by the value of n, which means by which of the three quantum states that is distributed.

We use the notation $|+--\rangle = |+\rangle \otimes |-\rangle \otimes |-\rangle$ etc., where the single spin states $|\pm\rangle$ are orthogonal states in a basis referred to as *basis I*. The three spinning particles are distributed to A, B and C, one particle to each of them, with the the first state in the tensor product corresponding to the spin sent to A, the second one to B and the third one to C. We assume the three-spin state is preserved under this distribution.

Each person in the receiving party can make (spin) measurements on the spinning particle he/she receives. The three can also communicate over a classical channel, which means that they can correlate their measurements and also compare the results of the measurements. They have, however no quantum channel available for communication. This means that all the observables that are available for measurements by the receiving party are of product form.

a) Determine the reduced density operator of A, and explain why, for any measurement he/she performs on his particle, no information can be extracted about which of the three spin states $|\psi_n\rangle$ is distributed. Also show that if A, B and C all make their spin measurements in *basis I*, even if they communicate their measured results, these cannot make any distinction between the three values of n.

Next, consider the situation where A and B are not able to communicate with C. They decide to perform measurements on the two spins they have received, and to make a probabilistic evaluation for the different values of n, based on the measured results. In order to do so they decide both to make their spin measurements in a rotated basis, which we refer to as *basis II*. The vectors in this basis are

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,, \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{7}$$

The possible outcomes of the measurements they list with numbers $k = 1, 2, 3, 4$, with the correspondence

$$k = 1:\ (0\,,0)\,, \quad k = 2:\ (0\,,1)\,, \quad k = 3:\ (1\,,0)\,, \quad k = 4:\ (1\,,1) \tag{8}$$

We refer to the corresponding states as $|\phi_k\rangle$, with $|\phi_1\rangle = |00\rangle = |0\rangle \otimes |0\rangle$, etc.

Before they do the measurements they evaluate for each three-spin state $|\psi_n\rangle$ the probabilities for the different measurement results (labeled by k). These probabilities are referred to as $p(k|n)$.

b) Find the reduced density operator $\hat{\rho}_n^{AB}$ and determine the probabilities $p(k|n)$ for different values of k and n. It is sufficient, due to repetitions of results, to consider $n = 0, 1$ and $k = 1, 2$.

Do you, in particular, see a reason why the probabilities are the same for $n = 1$ and $n = 2$, for all k?

c) Assume now that A and B perform their measurements, with the result labeled by k. The probability for the state to be $|\psi_n\rangle$, under the condition that the measured result is k, we denote by $\bar{p}(n|k)$. Under the assumption that all spin states $|\psi_n\rangle$ are equally probable until the result of the measurement is known, statistics theory gives us the following relation

$$\bar{p}(n|k) = \frac{p(k|n)}{p(k)} \tag{9}$$

with $p(k)$ as a normalization factor. Determine $p(k)$ and the probability $\bar{p}(n|k)$ for each n in the case $k = 1 : \ (0\ ,0)$. What is most probably the message that has been distributed?

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i: FYS 4110/9110 Ikke-relativistisk kvantemekanikk
Eksamensdag: Mandag 9. desember, 2013
Tid for eksamen: 4 timer, fra kl. 14:30
Oppgavesettet består av 3 oppgaver på 3 sider
Tillatte hjelpemidler: Godkjent kalkulator
Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

OPPGAVE 1
Tidsutvikling i et to-nivåsystem

Hamiltonoperatoren for et isolert to-nivåsystem (betegnet A) har formen $\hat{H}_0 = (1/2)\hbar\omega\,\sigma_z$, med σ_z som den diagonale Paulimatrisen. Vi betegner den normerte grunntilstandsvektoren som $|g\rangle$ og den eksiterte tilstanden som $|e\rangle$. Systemet er i realiteten koblet til et strålingsfelt (betegnet S), og den eksiterte tilstanden vil derfor henfalle til grunntilstanden under utsendelse av et strålingskvant. Vi lar $\hat{\rho}$ betegne den reduserte tetthetsoperatoren til delsystem A. Med god tilnærmelse kan tidsutviklingen av denne beskrives av den såkalte Lindbladligningen, her på formen

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[H_0, \hat{\rho}\right] - \frac{1}{2}\gamma\left[\hat{\alpha}^\dagger\hat{\alpha}\hat{\rho} + \hat{\rho}\hat{\alpha}^\dagger\hat{\alpha} - 2\hat{\alpha}\hat{\rho}\hat{\alpha}^\dagger\right] \tag{1}$$

med γ som henfallsraten for overgangen $|e\rangle \to |g\rangle$, $\hat{\alpha} = |g\rangle\langle e|$ og $\hat{\alpha}^\dagger = |e\rangle\langle g|$.

På matriseform, i basis $\{|e\rangle, |g\rangle\}$, skriver vi tetthetsoperatoren $\hat{\rho}$ som

$$\hat{\rho} = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{2}$$

med p_e som sannsynligheten for å finne systemet i tilstand $|e\rangle$ og p_g som sannsynligheten for å finne det i tilstand $|g\rangle$.

a) Anta først at to-nivåsystemet ved tiden $t = 0$ er i tilstanden $\hat{\rho} = |e\rangle\langle e|$. Vis ved bruk av ligning (1) at sannsynligheten p_e avtar eksponensielt, med γ som henfallsrate, mens total sannsynlighet $p_e + p_g$ er bevart.

b) Anta så en annen initialtilstand hvor to-nivåsystemet ved $t = 0$ er i den rene tilstanden $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Bestem den tidsavhengige tetthetsmatrisen $\hat{\rho}(t)$ med denne initialtilstanden.

c) Tetthetsoperatoren for system A kan alternativt uttrykkes ved Paulimatrisene, som $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$. Bestem funksjonen $r^2(t)$ i de to tilfellene ovenfor og vis at den i begge tilfeller har minimum for $t = (1/\gamma)\ln 2$. Hva blir minimalverdien til r i de to tilfellene? Gi en kommentar om hva dette sier om sammenfiltringen mellom systemene A og S. (Vi forutsetter at det fulle systemet A+S hele tiden er i en ren tilstand.)

OPPGAVE 2
Tre partikler i en sammenfiltret tilstand

Tre partikler med halvtallig spinn, som vi referer til som $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, befinner seg i en sammensatt spinntilstand

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle + |ddd\rangle) \tag{3}$$

hvor $|uuu\rangle = |u\rangle_{\mathcal{A}} \otimes |u\rangle_{\mathcal{B}} \otimes |u\rangle_{\mathcal{C}}$, er tensorproduktet av *spinn-opp (u)* langs z-aksen for alle tre partiklene, mens $|ddd\rangle = |d\rangle_{\mathcal{A}} \otimes |d\rangle_{\mathcal{B}} \otimes |d\rangle_{\mathcal{C}}$ er tensorprodukttilstanden svarende til *spinn-ned (d)* for alle tre partiklene. Vi antar at posisjonskoordinatene er helt frakoblet spinnkoordinatene og at spinnet derfor kan studeres separat. Det er ingen vekselvirkning mellom partiklene, og tilstanden (3) er derfor uendret så lenge det ikke måles på noen av spinnene.

Vi studerer i det følgende spinnsystemet som todelt, svarende til en oppsplitting $\mathcal{ABC} = \mathcal{A} + \mathcal{BC}$, slik at spinn $\mathcal{A}$ definerer det ene undersystemet og de to andre spinnene, $\mathcal{B}$ og $\mathcal{C}$, definerer det andre undersystemet.

a) Bestem de reduserte tetthetsoperatorene $\hat{\rho}_{\mathcal{A}}$ og $\hat{\rho}_{\mathcal{BC}}$ for de to delsystemene, og sammenfiltringsentropien til det sammensatte systemet. Hva menes med at de to delsystemene i denne tilstanden er *maksimalt* sammenfiltret? Delsystemet $\mathcal{BC}$ kan videre tenkes sammensatt av undersystemene $\mathcal{B}$ og $\mathcal{C}$. Hva sier tetthetsoperatoren $\hat{\rho}_{\mathcal{BC}}$ om sammenfiltring mellom disse to.

b) Ved et gitt tidspunkt blir en spinnmåling utført på partikkel $\mathcal{A}$ som bestemmer spinnkomponenten langs x-aksen som *spinn-opp* langs denne aksen. Denne informasjonen medfører at tetthetsoperatoren til systemet $\mathcal{BC}$ blir endret. Hva blir den nye reduserte tetthetsoperator $\hat{\rho}'_{\mathcal{BC}}$? Har måling på spinnet til partikkel $\mathcal{A}$ forandret sammenfiltringen mellom $\mathcal{B}$ og $\mathcal{C}$?

Vi minner om følgende: Med $|f\rangle$ som spinn-opp langs x-aksen og $|b\rangle$ som spinn-ned langs samme akse har vi relasjonene

$$|u\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle)\,, \quad |d\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle) \tag{4}$$

c) Anta at tre-spinnsystemet igjen befinner seg i tilstanden (3). Denne gangen måles spinnet til $\mathcal{A}$ langs en akse i xz-planet, som er rotert med vinkelen θ i forhold til z-aksen. Anta også at i dette tilfellet er måleresultatet *spinn-opp*. Finn hvordan måleresultatet nå påvirker tetthetsoperatoren for systemet $\mathcal{BC}$, og bestem sammenfiltringsentropien for sammensetningen $\mathcal{B} + \mathcal{C}$, som funksjon av vinkelen θ.

For de roterte spinntilstandene gjelder

$$\begin{aligned} |\theta, +\rangle &= \cos(\theta/2)|u\rangle + \sin(\theta/2)|d\rangle \qquad \text{(spinn opp)} \\ |\theta, -\rangle &= -\sin(\theta/2)|u\rangle + \cos(\theta/2)|d\rangle \quad \text{(spinn ned)} \end{aligned} \tag{5}$$

der $\theta = 0$ svarer til kvantisert spinn langs z-aksen og $\theta = \pi/2$ til kvantisert spinn langs x-aksen.

OPPGAVE 3
Spinnflipp-stråling

Vi studerer i denne oppgaven overgang mellom to spinntilstander for et elektron i et ytre magnetfelt som er rettet langs z-aksen, $\mathbf{B} = B\mathbf{e}_z$. (Merk: vi benytter her $\mathbf{e}_x$, $\mathbf{e}_y$ og $\mathbf{e}_z$ som enhetsvektorer langs x-, y- og z-aksen.) Hamiltonoperatoren skrives som $\hat{H} = \hat{H}_0 + \hat{H}_1$, hvor $\hat{H}_0$ svarer til den magnetiske dipolenergien i det ytre magnetfeltet, mens $\hat{H}_1$ beskriver koblingen mellom spinnet og strålingsfeltet. Vi har

$$\hat{H}_0 = \frac{1}{2}\omega_B \sigma_z \,, \qquad \omega_B = -\frac{eB}{m} \tag{6}$$

med e som elektronladningen og m som elektronmassen. Frekvensen ω_B regnes som positiv.

Matriseelementet til spinnvekselvirkningen $\hat{H}_1$, ved emisjon av et foton, er i dipoltilnærmelsen

$$\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle = i\frac{e\hbar}{2m}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\,(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}a})\cdot\boldsymbol{\sigma}_{BA} \tag{7}$$

hvor $|A\rangle$ er den eksiterte spinntilstanden (spinn-opp) og $|B\rangle$ er grunntilstanden (spinn-ned). Videre er $\mathbf{k}$ bølgetallsvektoren, $\boldsymbol{\epsilon}_{\mathbf{k}a}$ en polarisasjonsvektor og $\omega = c\,k$ er vinkelfrekvensen til det emitterte fotonet. V er et normeringsvolum for den elektromagnetiske strålingen og $\boldsymbol{\sigma}_{AB}$ er matriseelementet til Paulimatrisen $\boldsymbol{\sigma} = \sigma_x\,\mathbf{e}_x + \sigma_y\,\mathbf{e}_y + \sigma_z\,\mathbf{e}_z$ mellom de to spinntilstandene.

a) Til første orden i perturbasjonsteori vil vinkelavhengigheten til det kvadrerte matriseelementet (summert over polarisasjonsindeksen) $\sum_a |\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle|^2$, bestemme sannsynlighetsfordelingen for retningen til fotonet, $p(\phi,\theta)$, hvor (ϕ,θ) er polarvinklene til bølgevektoren $\mathbf{k}$. Bestem $p(\phi,\theta)$ fra uttrykket ovenfor. Vi minner om at ved summasjon over polarisasjonsretningene har vi $\sum\limits_a |\boldsymbol{\epsilon}_{\mathbf{k}a}\cdot\mathbf{b}|^2 = |\mathbf{b}|^2 - |\mathbf{b}\cdot\frac{\mathbf{k}}{k}|^2$ for en vilkårlig vektor $\mathbf{b}$. Normeringen av sannsynlighetsfordelingen er $\int d\phi \int d\theta \sin\theta\, p(\phi,\theta) = 1$.

b) Det kvadrerte matriseelementet (uten sum over a) bestemmer også, for gitt $\mathbf{k}$, sannsynlighetsfordelingen over polarisasjonsretningen til fotonet. Anta at en fotondetektor registrerer fotoner utsendt langs x-aksen ($\mathbf{k} = k\mathbf{e}_x$), med polarisasjonsretning $\boldsymbol{\epsilon}(\alpha) = \cos\alpha\,\mathbf{e}_y + \sin\alpha\,\mathbf{e}_z$. Hva er sannsynligheten $p(\alpha)$ for å detektere det emitterte fotonet? Anta her at sannsynligetsfordelingen er normert slik at summen over to ortogonale retninger er, $p(\alpha) + p(\alpha + \pi/2) = 1$. Hva sier resultatet om polarisasjonen til det emitterte fotonet?

Vi minner om standardformen på Paulimatrisene,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i: FYS 4110 / 9110 Ikke-relativistisk kvantemekanikk
Eksamensdag: Mandag 8. desember, 2014
Tid for eksamen: 4 timer, fra kl. 14:30
Oppgavesettet består av x oppgaver på y sider
Tillatte hjelpemidler: Godkjent kalkulator
Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

1 Entanglement in a two-spin system

We consider a composite quantum system consisting of two spin-half systems, A and B. The relevant states are restricted to the two-dimensional subspace spanned by the two (orthogonal) Bell states

$$|1\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)\,, \quad |2\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \tag{1}$$

where we use the notation $|+-\rangle = |+\rangle \otimes |-\rangle$, with $|\pm\rangle$ refering to the two eigenstates of σ_z.

Consider first (Case I) a linear superposition of the two state vectors, of the form

$$|\psi(x)\rangle = \cos x\,|1\rangle + \sin x\,|2\rangle\,, \quad 0 \leq x \leq \frac{\pi}{2} \tag{2}$$

The corresponding density operator we denote by $\hat{\rho}_I(x) = |\psi(x)\rangle\langle\psi(x)|$.

a) Determine the reduced density operators $\hat{\rho}_{IA}(x)$ and $\hat{\rho}_{IB}(x)$ of the two spins and the corresponding entropies $S_{IA}(x)$ and $S_{IB}(x)$. Characterize the entanglement of the two spins for the special values $x = 0$, $\pi/4$, and $\pi/2$.

Consider next (Case II) the following linear combination of the density operators of the two Bell states,

$$\hat{\rho}_{II}(x) = \cos^2 x\,|1\rangle\langle 1| + \sin^2 x\,|2\rangle\langle 2|\,, \quad 0 \leq x \leq \frac{\pi}{2} \tag{3}$$

b) What is the von Neuman entropy of this state? Find the reduced density operators $\hat{\rho}_{IIA}(x)$ and $\hat{\rho}_{IIB}(x)$, and the corresponding entropies $S_{IIA}(x)$, and $S_{IIB}(x)$. Characterize also here the states of the full system for $x = 0$, $\pi/4$, and $\pi/2$.

For a composite quantum system in pure quantum state, the degree of entanglement is expressed by the von Neumann entropy of one of its subsystems. When the system is in a mixed state we do not have a general, universally accepted, measure for the degree of entanglement. However, for a classical, statistical system we have the following inequality for the entropy of the full systems and its subsystem,

$$\Delta \equiv S - \max\{S_A, S_B\} \geq 0 \tag{4}$$

The breaking of this inequality in quantum system therefore indicates that the two subsystems are entangled.

c) Show that in the two cases I and II the functions $\Delta_I(x)$ and $\Delta_{II}(x)$ are negative for all x, except for one value of x.

OPPGAVE 2
Radiation damping

A charged particle is oscillating in a one-dimensional harmonic oscillator potential. It emits electric dipole radiation, with the rate for transition between an initial state i and a final state f given by the radiation formula

$$w_{fi} = \frac{4\alpha}{3c^2}\omega_{fi}^3|x_{fi}|^2 \tag{5}$$

where α is the fine structure constant, $\hbar\omega_{fi}$ is the energy radiated in the transition, and c is the speed of light. x is the position coordinate of the particle, which is related to the raising and lowering operators of the harmonic oscillator by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \tag{6}$$

with m as the mass of the particle.

a) Show that the non-vanishing transition rates are of the form

$$w_{n-1,n} = \gamma n \tag{7}$$

with $n = 0, 1, 2, ...$ as referring to the energy levels of the harmonic oscillator, and γ as a constant decay parameter. Detemine γ.

The effect of the radiation on the state of the oscillating particle is described by the Lindblad equation in the following way

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[H_0, \hat{\rho}\right] - \frac{1}{2}\gamma\left[\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right] \tag{8}$$

with $\hat{\rho}$ as the density operator of the particle and H_0 as the harmonic oscillator Hamiltonian, without decay.

b) In the following we focus on the diagonal terms of the density matrix, $p_n = \rho_{nn} = \langle n|\hat{\rho}|n\rangle$, which define the occupation probabilities of the energy eigenstates. Show that they satisfy the equation

$$\frac{dp_n}{dt} = -\gamma(np_n - (n+1)p_{n+1}) \tag{9}$$

Explain why this is consistent with the expression (7) for the transition rate $w_{n-1,n}$.

c) Show that Eq. (9) implies that the expectation value of the excitation energy

$$E = \langle H_0 \rangle - \frac{1}{2}\hbar\omega \tag{10}$$

decays exponentially with time.

OPPGAVE 3
A state in thermal equilibrium

A quantum state in thermal equilibrium is described by the density operator

$$\hat{\rho}(\beta) = N(\beta)e^{-\beta\hat{H}} = N(\beta)\sum_n e^{-\beta E_n}|n\rangle\langle n| \tag{11}$$

with $\hat{H}$ as the Hamiltonian, E_n as the corresponding energy eigenvalues, and $N(\beta)$ as a normalization factor. The parameter β is related to the temperature T by $\beta = 1/(k_B T)$, with k_B as Boltzmann's constant.

a) Show that the expectation value for the energy can be expressed in terms of $N(\beta)$ as

$$E(\beta) = \frac{d}{d\beta}\ln N(\beta) \tag{12}$$

and find a similar expression for the von Neumann entropy $S(\beta) = \mathrm{Tr}[\hat{\rho}(\beta)\ln\hat{\rho}(\beta)]$. (Use here the natural logarithm in the definition of S.)

b) For a two-level system, with Hamiltonian $\hat{H} = (\epsilon/2)\,\sigma_z$, determine the functions $N(\beta)$, $E(\beta)$ and $S(\beta)$, and make a sketch of the expectation value of the energy E as function of the temperature T.

c) Find the density operator expressed in the form $\hat{\rho} = (1/2)(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$. Determine $\mathbf{r}$ as a function of β and relate this to the results in b).

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS4110/9110 Modern Quantum Mechanics
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Permitted materials: Approved calculator
Angell and Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference. Noen engelske ord er oversatt etter hver oppgave.

Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1
Two spin-half systems

A quantum system is composed of two interacting spin-half systems, referred to as system A and B. The Hamiltonian has the form

$$\hat{H} = \frac{1}{2}\hbar\omega(\,\sigma_z \otimes \mathbb{1} - \,\mathbb{1} \otimes \sigma_z) + \hbar\lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

where σ_z og $\sigma_\pm$ are Pauli matrices, with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hbar\omega$ giving the splitting between the two energy levels of each of the two spins, and with λ as a coupling parameter. The two factors of the tensor product refer to each of the two spin systems, with A corresponding to the first and B as the second factor. It is convenient to introduce new parameters by $\omega = a\cos\theta$ and $\lambda = a\sin\theta$, with $-\pi/2 < \theta \leq \pi/2$. We further use $|\pm\rangle$ as notation for the eigenstates of σ_z. In the following we use the tensor products of these states as basis for the Hilbert space of the composite system.

a) Show that only the product states $|+-\rangle = |+\rangle \otimes |-\rangle$ and $|-+\rangle = |+\rangle \otimes |-\rangle$ are mixed by the λ term in the Hamiltonian, and give the expression for the Hamiltonian as a 2x2 matrix, in the subspace spanned by $|+-\rangle$ and $|-+\rangle$.

b) Find the energy eigenvalues, and the energy eigenstates, expressed in terms of a and θ.

c) Determine, for the energy eigenstates, the density operator of the full system and the reduced density operators of the two subsystems, and detemine the entanglement entropy of the eigenstates as functions of θ. What are the minimum and maximum values of the entropy functions? Make a comparison with the maximal possible value of the entanglement entropy of the two-spin system.

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density operator = tetthetsoperator
entanglement = sammenfiltring

PROBLEM 2
A driven harmonic oscillator

A quantum mechanical, driven harmonic oscillator is described by the following Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \tag{2}$$

where $\hat{a}$ og $\hat{a}^\dagger$ satisfy the standard commutation relations for lowering and raising operators, and where ω_0, ω og λ are three constants.

a) As a reminder, Heisenberg's equation of motion has the form

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar}\left[H, \hat{A}\right] + \frac{\partial}{\partial t}\hat{A} \tag{3}$$

for any given observable $\hat{A}$. Apply this to the operator $\hat{a}_H$, which is the operator $\hat{a}$ transformed to the Heisenberg picture, and show that it satisfies an equation of the form

$$\frac{d^2\hat{a}_H}{dt^2} + \omega_0^2\hat{a}_H = Ce^{-i\omega t}\,\mathbb{1} \tag{4}$$

with C as a constant. Determine C.

b) Equation (4) can be solved as a linear differential equation, to give

$$\hat{a}_H(t) = \hat{a}\,e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\,\mathbb{1} \tag{5}$$

Show that (5) is a solution of (4) and determine the constant D.

c) A coherent state is defined as an eigenstate of the lowering operator $\hat{a}$,

$$\hat{a}|z\rangle = z|z\rangle \tag{6}$$

Assume that the oscillator, at time $t = 0$, is in the ground state for the λ-independent part of the Hamiltonian, that is

$$|\psi(0)\rangle = |0\rangle\,, \qquad \hat{a}|0\rangle = 0 \tag{7}$$

Show that, during the time evolution (in the Schrödinger picture), it will continue as a coherent state, so that

$$\hat{a}|\psi(t)\rangle = z(t)|\psi(t)\rangle \tag{8}$$

with $z(t)$ as a complex-valued function of time.

Find the function $z(t)$, and compare the time evolution of the real part $x(t) = (z(t)+z(t)^*)/2$ with the motion of the corresponding classical driven harmonic oscillator.

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driven harmonic oscillator = tvungen harmonisk oscillator
coherent state = koherent tilstand

PROBLEM 3
Atom and photon in an optical microcavity

An atom is contained in an optical microcavity, with the energy difference between two of the atomic levels matching exactly the frequency of one of the electromagnetic cavity modes. A simplified description of the photon-atom system has the form of a two-level system coupled to a single electromagnetic mode. The Hamiltonian then takes the form

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{9}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are photon creation and annihilation operators, and σ_z and $\sigma_\pm$ are Pauli matrices with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. These operators act between the two atomic levels with the upper and lower energy levels corresponding to the eigenvalues $+1$ and -1 respectively of σ_z. We refer in the following to $|\pm, n\rangle = |\pm\rangle \otimes |n\rangle$ as product states of the composite system, with $|\pm\rangle$ as the upper/lower atomic levels and $|n\rangle$ as the photon number states of the cavity mode.

a) Assume a single photon is introduced in the cavity at time $t = 0$ while the atom is in its ground state. Show that the atom-photon state will subsequently oscillate in the following way

$$|\psi(t)\rangle = e^{i\epsilon t}(\cos\Omega t\,|-, 1\rangle - i\sin(\Omega t)|+, 0\rangle) \tag{10}$$

and find Ω and ϵ expressed in terms of ω and λ.

To take into account leakage of photons from the cavity, we turn to a description of the time evolution in terms of the density operator. It is assumed to satisfy the Lindblad equation,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H, \hat{\rho}] - \frac{1}{2}\gamma\left[\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right] \tag{11}$$

where γ is the escape rate for photons from the cavity.

b) The probability for finding the atom in the ground state with no photon in the cavity is $p_g = \langle -, 0|\hat{\rho}|-, 0\rangle$. Assume that there is initially a non-vanishing probability for a photon being present in the cavity. Show that that this will result in an increase in p_g with time, which is consistent with the expectation that the photon will escape from the cavity.

c) Assuming there is no contribution to $\hat{\rho}$ from higher excited states than $|-, 1\rangle$ and $|+, 0\rangle$, show that a closed set of coupled differential equations for the three variables $p_1 = \langle -, 1|\hat{\rho}|-, 1\rangle$, $p_0 = \langle +, 0|\hat{\rho}|+, 0\rangle$ and $b = \mathrm{Im}\langle -, 1|\hat{\rho}|+, 0\rangle$ can be derived from the Lindblad equation.

Without solving the equations, give a qualitative description of what the expected time evolution will be with the same initial condition as in a).

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cavity mode = kavitetsmode, hulromsmode

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS4110/9110 Modern Quantum Mechanics
Day of exam: Monday 5 December 2016
Exam hours: 4 hours, beginning at 14:30
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Permitted materials: Approved calculator
Angell and Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference. *Note: This paper is also available in norsk bokmål and nynorsk.*

Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1
Spin-half particle in a harmonic oscillator potential
A spin-half particle is moving in a one-dimensional harmonic oscillator potential (in the x-direction) under the influence of a constant magnetic field (in the z-direction). The Hamiltonian is

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{1}{2}\hbar\omega_1\sigma_z + \lambda\hbar(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{1}$$

where the first term is the harmonic oscillator part, with ω_0 as the oscillator angular frequency, and the second term is the spin energy due to the magnetic field, with ω_1 as the angular spin precession frequency. The third term is a coupling term between the spin and the position coordinates of the particle, with λ as a coupling parameter. The spin flip operators are defined as $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hat{a}$, $\hat{a}^\dagger$ are the standard lowering and raising operators of the harmonic oscillator, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.

When $\lambda = 0$, the spin and position of the particle are uncoupled and the energy eigenstates are $|n, m\rangle$, with $n = 0, 1, 2, ...$ as the harmonic oscillator quantum number and $m = \pm 1$ as the spin quantum number, corresponding to spin up/down along the z-axis. When $\lambda \neq 0$, the unperturbed eigenstates will pairwise be coupled by the Hamiltonian, so that $|n, +1\rangle$ is coupled to $|n+1, -1\rangle$. (The state $|0, -\rangle$ is an exception; it is not affected by the coupling term and remains the non-degenerate ground state also for $\lambda \neq 0$.)

a) Consider the two-dimensional subspace spanned by the basis vectors $|0, +1\rangle$ and $|1, -1\rangle$. Show that in this space, and in the given basis, the Hamiltonian takes the form of a 2x2 matrix

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \hbar\epsilon\mathbb{1} \tag{2}$$

with $\mathbb{1}$ as the 2x2 identity matrix. Determine Δ, $\cos\theta$, $\sin\theta$ and ϵ.

b) Find the energies and eigenstates of H in the two-dimensional subspace, expressed as functions of Δ, θ and ϵ.

c) The basis vectors $|n, m\rangle$ can be regarded as tensor products of position and spin vectors, $|n, m\rangle = |n\rangle \otimes |m\rangle$. The two eigenstates found under b) will be entangled with respect to the position and spin variables. Determine the entanglement entropy as function of θ. What value for θ gives the least and what gives the greatest entanglement?

PROBLEM 2
Coupled harmonic oscillators

Two harmonic oscillators, referred to as $\mathcal{A}$ and $\mathcal{B}$, form a composite quantum mechanical system. The Hamiltonian of the system has the form

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + \mathbb{1}) + \hbar\lambda(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) \tag{3}$$

with $(\hat{a}, \hat{a}^\dagger)$ as lowering and raising operators for $\mathcal{A}$ and $(\hat{b}, \hat{b}^\dagger)$ as corresponding operators for $\mathcal{B}$, while ω and λ are real valued constants.

a) Show that the Hamiltonoperator can be expressed in diagonal form as

$$\hat{H} = \hbar\omega_c\, \hat{c}^\dagger \hat{c} + \hbar\omega_d\, \hat{d}^\dagger \hat{d} + \hbar\omega\mathbb{1} \tag{4}$$

where c and d are linear combinations of a and b,

$$\hat{c} = \mu\, \hat{a} + \nu\, \hat{b}\,, \quad \hat{d} = -\nu\, \hat{a} + \mu\, \hat{b} \tag{5}$$

with μ and ν as real constants satisfying $\mu^2 + \nu^2 = 1$, and determine the new parameters μ, ν, ω_c, and ω_d, expressed in terms of ω og λ. (The same type of expressions are found for the harmitian conjugate operators $\hat{c}^\dagger$ og $\hat{d}^\dagger$.) Check that the new operators $\hat{c}$ and $\hat{d}$ satisfy the same set of harmonic oscillator commutation relations as $\hat{a}$ and $\hat{b}$. It is sufficient to show

$$\left[\hat{c}, \hat{c}^\dagger\right] = \left[\hat{d}, \hat{d}^\dagger\right] = \mathbb{1}\,, \quad \left[\hat{c}, \hat{d}^\dagger\right] = 0 \tag{6}$$

b) Assume that the state $|\psi(0)\rangle$ of the composite system, at time $t = 0$, is a coherent state when expressed in terms of the new variables,

$$\hat{c}\,|\psi(0)\rangle = z_{c0}|\psi(0)\rangle\,, \quad \hat{d}\,|\psi(0)\rangle = z_{d0}|\psi(0)\rangle \tag{7}$$

Also at a later time the state $|\psi(t)\rangle$ will be a coherent state for both $\hat{c}$ og $\hat{d}$, with eigenvalues

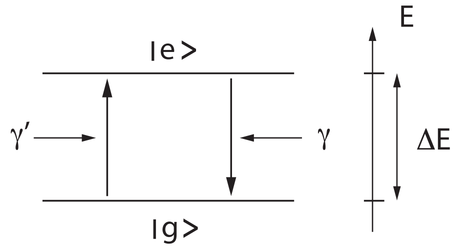
$$z_c(t) = e^{-i\omega_c t} z_{c0}\,, \quad z_d(t) = e^{-i\omega_d t} z_{d0} \tag{8}$$

Show this for $z_c(t)$. (The expression for $z_d(t)$ follows in the same way, and is therefore not needed to be shown.)

c) Show that the state $|\psi(t)\rangle$ is a coherent state also for the original harmonic oscillator operators $\hat{a}$ og $\hat{b}$, and find the eigenvalues $z_a(t)$ and $z_b(t)$ expressed in terms of $z_{a0}$and z_{b0}.

PROBLEM 3
Two-level system in a heat bath

We consider a two-level system, with $|g\rangle$ as the ground state and $|e\rangle$ as the excited state of the Hamiltonian $\hat{H}_0$. The energy difference between the corresponding two energy levels is ΔE.



The system interacts weakly with a heat bath with temperature T. Energy can flow both ways, with γ as the rate for emission of energy to the heat bath in the transition $|e\rangle \to |g\rangle$ and γ' as the rate for absorption of energy in the transition $|g\rangle \to |e\rangle$. The situation is illustrated in the figure.

The temperature T of the heat bath and the energy gap ΔE determine the ratio between γ' and γ,

$$\gamma' = \gamma\, e^{-\Delta E/kT} \tag{9}$$

where k is the Boltzman constant. Both transitions, corresponding to γ and γ', contribute to the time evolution of the density operator of the two-level system. This is expressed by the Lindblad equation in the following way,

$$\begin{aligned}\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] \;&-\; \frac{1}{2}\gamma(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e| - 2|g\rangle\langle e|\hat{\rho}|e\rangle\langle g|) \\ &-\; \frac{1}{2}\gamma'(|g\rangle\langle g|\hat{\rho} + \hat{\rho}|g\rangle\langle g| - 2|e\rangle\langle g|\hat{\rho}|g\rangle\langle e|)\end{aligned} \tag{10}$$

The 2×2 matrix form of $\hat{\rho}$, in the basis $\{|g\rangle, |e\rangle\}$, we write as

$$\rho = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{11}$$

with p_e interpreted as the probability of occupation of the excited level and p_g as the probability of occupation of the ground state.

a) Find from equation (10) expressions for the time derivatives of p_e, p_g and b, and check that they are consistent with preservation of total probability, $p_e + p_g$.

b) The conditions for $\hat{\rho}$ to be a density operator give restrictions on the matrix elements in (11). What are these?

c) Assume first that the two-level system and the heat bath are in thermal equilibrium, and the density matrix (11) therefore is time independent. Determine the values of variables p_e, p_g and b in this case.

d) Consider next the situation with initial values $p_g = 1, p_e = 0$. Determine the time evolution of the occupation probabilities towards thermal equilibrium. What happens in the limits $T \to 0$ and $T \to \infty$?

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

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This examination paper consists of 3 pages	
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	Angell and Lian: Størrelser og enheter i fysikken
	Rottmann: Matematisk formelsamling

Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1

Two interacting Two Level Systems

We have two interacting Two Level Systems, which we call systems A and B, with their corresponding sets of Pauli matrices σ_i^A and σ_i^B. The Hamiltonian is the following:

$$H = \frac{1}{2}\hbar g \sigma_z^A \otimes \sigma_z^B$$

where g is the interaction strength. Here we use a representation where for each system $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

a) Find the time evolution operator $U(t) = e^{-\frac{i}{\hbar}Ht}$ in the form of a 4×4 matrix.

b) Assume that at time $t = 0$ the two systems are in a product state

$$|\psi(0)\rangle = |\psi^A(0)\rangle \otimes |\psi^B(0)\rangle$$

with

$$|\psi^A(0)\rangle = a|0\rangle + b|1\rangle \qquad \text{and} \qquad |\psi^B(0)\rangle = c|0\rangle + d|1\rangle.$$

with $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$. Find the reduced density matrices for systems A and B as functions of time.

c) We define the Bloch vectors of A and B as $\mathbf{m}$ and $\mathbf{n}$, respectively, so that

$$\rho^A = \frac{1}{2}\left(\mathbb{1} + \mathbf{m}\cdot\sigma^A\right) \qquad \text{and} \qquad \rho^B = \frac{1}{2}\left(\mathbb{1} + \mathbf{n}\cdot\sigma^B\right)$$

Consider now the special case $a = b = \frac{1}{\sqrt{2}}$. Find the Bloch vector $\mathbf{m}$ for system A and show that as a function of time it is describing an ellipse in the xy-plane.

d) For given initial values c and d for system B and still $a = b = \frac{1}{\sqrt{2}}$, find the maximal value of the entanglement entropy, and show that it depends only in the component n_z of the Bloch vector $\mathbf{n}$ for system B.

PROBLEM 2

Squeezed states of the harmonic oscillator

We have in the lectures studied coherent states of the harmonic oscillator as examples of minimal uncertainty states. Here we will consider a related class of minimal uncertainty states called squeezed states. We define the squeeze operator

$$S(\zeta) = e^{-\frac{1}{2}(\zeta\hat{a}^2 - \zeta^*\hat{a}^{\dagger 2})}$$

where ζ is a complex number and $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators of the harmonic oscillator. The squeezed vacuum state is defined as

$$|sq_\zeta\rangle = S(\zeta)|0\rangle$$

a) Show that the action of the squeeze operator on $\hat{a}$ and $\hat{a}^\dagger$ is given by

$$S^\dagger(\zeta)\hat{a}S(\zeta) = \cosh r\hat{a} + e^{-i\phi}\sinh r\hat{a}^\dagger$$
$$S^\dagger(\zeta)\hat{a}^\dagger S(\zeta) = \cosh r\hat{a}^\dagger + e^{i\phi}\sinh r\hat{a}$$

where $\zeta = re^{i\phi}$.

b) In the state $|sq_\zeta\rangle$, find the variance of the position and momentum operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \text{and} \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right).$$

That is, calculate

$$\Delta x^2 = \langle sq_\zeta|\hat{x}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{x}|sq_\zeta\rangle^2$$
$$\Delta p^2 = \langle sq_\zeta|\hat{p}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{p}|sq_\zeta\rangle^2$$

c) The Heisenberg uncertainty relation tells us that $\Delta x \Delta p \geq \frac{\hbar}{2}$ with equality only for minimal uncertainty states. Calculate the product $\Delta x \Delta p$ for the states $|sq_\zeta\rangle$ and show that for certain ϕ they are minimal uncertainty states.

d) For those ϕ which gives minimal uncertainty, compare Δx and Δp with the corresponding values in vacuum and describe what happens to the uncertainties.

e) For a general value of ϕ the state $|sq_\zeta\rangle$ is not of minimal uncertainty with respect to the operators $\hat{x}$ and $\hat{p}$. However, for any ϕ we can find transformed operators $\hat{x}_\phi$ and $\hat{p}_\phi$ satisfying the usual commutator relation $[\hat{x}_\phi, \hat{p}_\phi] = i\hbar$ and where $\Delta x_\phi \Delta p_\phi = \frac{\hbar}{2}$. Here Δx_ϕ and Δp_ϕ are defined by the same equations as Δx and Δp with $\hat{x}$ and $\hat{p}$ replaced by $\hat{x}_\phi$ and $\hat{p}_\phi$. Determine $\hat{x}_\phi$ and $\hat{p}_\phi$ expressed in terms of ϕ, $\hat{x}$ and $\hat{p}$.

We remind you of the general relation

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2}[B, [B, A]] + \cdots$$

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
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	Angell and Lian: Størrelser og enheter i fysikken
	Rottmann: Matematisk formelsamling

Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1

Pure and mixed states

a) Explain what is the difference between pure and mixed quantum states. How are they represented mathematically?

b) An ensemble of spin-$\frac{1}{2}$ particles are produced by some (to you) unknown procedure. You are informed that the particles will be either (ensemble A) in the state $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ or (ensemble B) in a random statistical mixture with 50% of the particles in the state $|\uparrow\rangle$ and 50% of the particles in the state $|\downarrow\rangle$. You are allowed to measure the spin of each particle along an axis of your choice (you do not have to choose the same axis for each particle). Describe an experiment which would reveal whether the particles are prepared in ensemble A or ensemble B. Explain what will be the probabilities of different measurement oucomes for both ensembles when using your measurement procedure.

c) Consider now a third enemble (ensemble C), where the particles are prepared in a random statistical mixture with 50% of the particles in the state $|\rightarrow\rangle$ and 50% of the particles in the state $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. Prove that we can not distiguish ensembles B and C by any measurements on the particles.

Instead of direct preparation as described above, we can prepare the ensembles B or C remotely by entanglement in the following way. Person 1 (the preparer) prepares an ensemble of pairs of entangled particles in the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. He keeps one particle from each pair to himself and sends the other particle from each pair to person 2 (you). By doing appropriate measurements on his particles, person 1 can now decide at a later point if he would like your particles to belong to ensemble B or C.

d) Which measurement should person 1 perform to generate ensemble B and which to generate ensemble C? Justify your answer.

e) Even if the ensembles B and C are indistiguishable by local measurements by person 2, as you showed in question c), they can be distiguished by the correlations between the measurement outcomes of persons 1 and 2. Explain which measurements person 2 should do, and how the difference between ensembles B and C are visible in the correlations. Assume that the pairs are labeled, so that we can compare the measurement oucomes for the two particles belonging to the same pair. What changes if person 1 decides to wait with his measurements until after person 2 makes the measurements, so that the two ensembles are not prepared until after they are measured.

PROBLEM 2
Coupled harmonic oscillators

Two identical harmonic oscillators, A and B, are coupled with a Hamiltonian

$$H = \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) + \hbar\lambda(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}). \tag{1}$$

Here $\hat{a}^\dagger$ and $\hat{a}$ are creation and annihilation operators for oscillator A and $\hat{b}^\dagger$ and $\hat{b}$ corresponding operators for oscillator B.

a) Show that the Hamiltonian can be expressed in diagonal form as

$$H = \hbar\omega_c\hat{c}^\dagger\hat{c} + \hbar\omega_d\hat{d}^\dagger\hat{d} \tag{2}$$

where $\hat{c}$ and $\hat{d}$ are linear combinations of $\hat{a}$ and $\hat{b}$

$$\hat{c} = \mu\hat{a} + \nu\hat{b}, \qquad \hat{d} = -\nu\hat{a} + \mu\hat{b} \tag{3}$$

where μ and ν are positive real constants satisfying $\mu^2 + \nu^2 = 1$. Determine the constants μ, ν, ω_c and ω_d in terms of ω and λ. Check that the operators $\hat{c}$ and $\hat{d}$ satisfy the usual harmonic oscillator commutation relations, and that the oscillators C and D are independent of each other (all operators for different oscillators commute).

b) We define the number operators for the original oscillators as $N_A = \hat{a}^\dagger\hat{a}$ and $N_B = \hat{b}^\dagger\hat{b}$. Assume that the initial state of the system is the first excited state of oscillator A. That is, the state $\hat{a}^\dagger|0\rangle$ where $|0\rangle$ is the ground state. Find the expectation values $\langle N_A\rangle$ and $\langle N_B\rangle$ as functions of time. Describe the result.

c) Calculate the entanglement entropy between oscillators A and B as a function of time. What is the maximal value of the entanglement entropy. At what times is the entropy zero and what is the state of the system at these times?

PROBLEM 3

Driven two-level system with damping

The Hamiltonian of an isolated two-level system is $H_0 = \frac{1}{2}\hbar\omega_0\sigma_z$. Let $|g\rangle$ be the ground state and $|e\rangle$ the excited state. The system is coupled to a radiation field, so that the excited state spontaneously will decay to the ground state, emitting a quantum of radiation (which could be photons, phonons or some other field excitation depending on the physical realization). This means that the density matrix ρ of the system will (to a good approximation) satisfy a Lindblad equation of the form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - \frac{1}{2}\gamma\left[\alpha^\dagger\alpha\rho + \rho\alpha^\dagger\alpha - 2\alpha\rho\alpha^\dagger\right] \tag{4}$$

where γ is the decay rate for the transition $|e\rangle \to |g\rangle$ and $\alpha = |g\rangle\langle e|$.

a) We parametrize the density matrix in the following way

$$\rho = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix}. \tag{5}$$

Derive the equations for $\dot{p}_e$, $\dot{p}_g$ and $\dot{b}$ and check that they are consistent with the conservation of total probability, $p_e + p_g = 1$.

b) Find the solution of the Lindblad equation if the initial state is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Calculate the Bloch vector as a function of time and describe its motion in the Bloch sphere (Reminder: The density matrix can be expressed as $\rho = \frac{1}{2}(1 + \mathbf{r}\cdot\sigma)$ where $\mathbf{r}$ is the Bloch vector).

We excite the two-level system by an external wave, which we assume is described by adding a time dependent driving term to the Hamiltonian, so that it takes the form

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \frac{1}{2}\hbar\omega_1(\cos\omega t\sigma_x + \sin\omega t\sigma_y). \tag{6}$$

c) We want to study the system in a reference frame rotating around the z-axis with the frequency ω of the external wave. That is, we define the state in the rotating frame as $|\psi'\rangle = T(t)|\psi\rangle$ where $T(t)$ is a time dependent unitary transformation. Determine the form of $T(t)$ and derive the form of the Lindblad equation in the rotating frame.

d) Find the stationary solution of the Lindblad equation in the rotating frame. Describe the result in the limiting cases of small and large ω_1. What quantity should ω_1 be compared to for the limits to apply?

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
Day of exam:	**5. December 2019**
Exam hours:	**9.00-13.00, 4 hours**
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Permitted materials:	**Approved electronic calculator.**
	Angell and Lian: Størrelser og enheter i fysikken
	Rottmann: Matematisk formelsamling

Language: The solutions may be written in Norwegian or English depending on your own preference.
Make sure that your copy of this examination paper is complete before answering.
All answers should be justified

PROBLEM 1
Three-spin entanglement

We have three spin-$\frac{1}{2}$ particles, A, B and C, in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(| \uparrow\downarrow\downarrow\rangle + | \downarrow\uparrow\downarrow\rangle + | \downarrow\downarrow\uparrow\rangle \right).$$

where $| \uparrow\rangle$ and $| \downarrow\rangle$ are the eigenstates of the spin in the z-direction. We consider the splitting of this system in two subsystems, one consisting of particle A and the other of particles B and C.

a) Find reduced density matrices for the two subsystems. Find the entanglement entropy.

b) We now make a measurement of the spin of particle A in the z-direction. What is the final state of the system after the measurement (give the answer for all possible measurement outcomes)? What is the entanglement entropy of the particles B and C after the measurement?

c) We now measure the spin of particle A in the x-direction instead. What is the entanglement entropy of the particles B and C after this measurement?

PROBLEM 2
Bloch-Siegert shift

We consider first an electron in a constant external magnetic field in the z-direction subject to a rotating field in the xy-plane. The Hamiltonian has the form

$$H = \frac{\hbar}{2}\omega_0\sigma_z + \frac{\hbar}{2}A(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y)$$

Here ω_0 is the natural precession frequency of the electron spin in the external field, A is the amplitude of the driving field, and ω its frequency.

a) Show that by changing to a reference frame rotating with the frequency ω of the driving field, the total field is constant in the rotating frame. From the Hamiltonian in the rotating frame, conclude that resonance (in the sense of largest amplitude Rabi oscillations of the spin state if the initial state is the ground state) will take place when $\omega = \omega_0$ irrespective of the driving amplitude A.

Now we replace the rotating field by one oscillating in the x-direction, which in many cases is more realistic. The Hamiltonian now reads

$$H = \frac{\hbar}{2}\omega_0\sigma_z + \frac{\hbar}{2}A\cos(\omega t)\sigma_x. \tag{1}$$

b) Show that using the same transformation as above, the Hamiltonian in the rotating frame will get an additional term which describes a field rotating at the frequency 2ω and give an explanation for why this happens. Explain why we in some cases to a good approximation can neglect this additional rotating component, and use the same Hamiltonian as we had for the rotating field also when the field is oscillating, which is usually called the rotating wave approximation.

We will now study how we can get more accurate results than what is obtained in the rotating wave approximation. To achieve this, we will start from the Hamiltonian (1), but instead of going to a rotating frame, we will make the transformation

$$|\psi'\rangle = e^{iS(t)}|\psi\rangle, \qquad S(t) = \frac{A}{2\omega}\xi\sin(\omega t)\,\sigma_x$$

where ξ is a parameter whose value we will choose later.

c) Show that the transformed Hamiltonian is

$$H' = \frac{\hbar}{2}\omega_0\left\{\cos\left[\frac{A}{\omega}\xi\sin(\omega t)\right]\sigma_z + \sin\left[\frac{A}{\omega}\xi\sin(\omega t)\right]\sigma_y\right\} + \frac{\hbar}{2}A(1-\xi)\cos(\omega t)\,\sigma_x.$$

Using the identities

$$
\begin{aligned}
\cos\left[\frac{A}{\omega}\xi\sin(\omega t)\right] &= J_0\left(\frac{A}{\omega}\xi\right) + 2\sum_{k=1}^{\infty} J_{2k}\left(\frac{A}{\omega}\xi\right)\cos(2k\omega t) \\
\sin\left[\frac{A}{\omega}\xi\sin(\omega t)\right] &= 2\sum_{k=0}^{\infty} J_{2k+1}\left(\frac{A}{\omega}\xi\right)\sin[(2k+1)\omega t]
\end{aligned}
$$

where $J_k(x)$ is the Bessel function of the first kind of order k, one can find that $H' = H_0' + H_1' + H_2'$ with

$$
\begin{aligned}
H_0' &= \frac{\hbar}{2}\omega_0 J_0\left(\frac{A}{\omega}\xi\right)\sigma_z \\
H_1' &= \hbar\omega_0\sin(\omega t)J_1\left(\frac{A}{\omega}\xi\right)\sigma_y + \frac{\hbar}{2}A(1-\xi)\cos(\omega t)\sigma_x \\
H_2' &= \hbar\omega_0\sum_{k=0}^{\infty}\left\{J_{2k}\left(\frac{A}{\omega}\xi\right)\cos(2k\omega t)\sigma_z + J_{2k+1}\left(\frac{A}{\omega}\xi\right)\sin[(2k+1)\omega t]\sigma_y\right\}.
\end{aligned}
$$

You do not have to derive this. All the terms in H_2' have higher frequencies than the typical dynamical frequencies of the state, and we will ignore H_2' and approximate $H' \approx H_0' + H_1'$. We will also choose ξ so that it satisfies the equation

$$
J_1\left(\frac{A}{\omega}\xi\right)\omega_0 = \frac{1}{2}A(1-\xi).
$$

d) Explain what is special about this choice of ξ and why this simplifies the problem. Show that the resonance condition for large amplitude Rabi oscillations now is

$$
\omega = \omega_0 J_0\left(\frac{A}{\omega}\xi\right). \tag{2}
$$

e) According to Eq. (2), the resonance frequency now depends on the amplitude, in contrast to the case of a rotating field studied in question a). Use the series expansions for the Bessel functions

$$
\begin{aligned}
J_0(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64} + \cdots \\
J_1(x) &= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} + \cdots
\end{aligned}
$$

to show that in the limit of a weak driving field, $A \to 0$, we recover the resonance at $\omega = \omega_0$ as we had using the rotating wave approximation, and find the lowest order in A correction to the resonance frequency for small A.

PROBLEM 3
Spinflip radiation

We will study the transition between the two spin states of an electron in an external magnetic field directed along the z-axis, $\mathbf{B} = b\mathbf{e}_z$. The Hamiltonian can be expressed as $H = H_0 + H_1$, where H_0 descibes the energy of a magnetic dipole in the external field, while H_1 describes the interaction with the radiation field. Then we have

$$H_0 = \frac{\hbar}{2}\omega_B \sigma_z, \qquad \omega_B = -\frac{eB}{m}$$

where m is the electron mass and e the electron charge (which is negavive so that $\omega_B > 0$). The matrix element of the interaction pat H_1 for the case of emission if a single photon is in the dipole approximation given by

$$\langle B, 1_{\mathbf{k}a}|H_1|A, 0\rangle = i\frac{e\hbar}{2m}\sqrt{\frac{\hbar}{2\omega V \epsilon_0}}(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}a}) \cdot \boldsymbol{\sigma}_{BA}$$

where $|A\rangle$ is the excited spin state (spin up) and $|B\rangle$ is the ground state (spin down). $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is the polarization vector and $\omega = ck$ is the angular frequency of the emitted photon. V is the normalization volume for the electromagnetic radiation and $\boldsymbol{\sigma}_{BA} = \langle B|\boldsymbol{\sigma}|A\rangle$ is the matrix element of the vector $\boldsymbol{\sigma}$ of the Pauli matrices.

a) To first order in perturbation theory, the angular dependency of the squared matrix element $|\langle B, 1_{\mathbf{k}a}|H_1|A, 0\rangle|^2$ will determine the probability distribution for the direction of the emitted photon, $p(\theta, \phi)$, where (θ, ϕ) are the polar coordinates for the wavevector $\mathbf{k}$. Determine $p(\theta, \phi)$ using the above expression for the matrix element. It may be useful to know that when summing over the polarization states we have $\sum_a |\boldsymbol{\epsilon}_{\mathbf{k}a} \cdot \mathbf{b}|^2 = |\mathbf{b}|^2 - |\mathbf{b} \cdot \frac{\mathbf{k}}{k}|^2$ for an arbitrary vector b. The probability distribution should be normalized as $\int d\phi \int d\theta \sin\theta p(\theta, \phi) = 1$.

b) The squared matrix element also determines, for a given $\mathbf{k}$, the probability distribution for the polarization direction of the photon. Assume that a photon detector is set to detect photons emitted in the x-direction and with polarization vector $\boldsymbol{\epsilon}(\alpha) = \cos\alpha\mathbf{e}_y + \sin\alpha\mathbf{e}_z$ (here $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors in the x- and y- directions). Find the probability distribution $p(\alpha)$ to detect the emitted photon as a function of the angle α.

c) To a good approximation, the probability to find the spin in the excited state decays exponentielly with time

$$P_A(t) = e^{-t/\tau}$$

where the lifetime τ is, to first order in the interaction, determined by the time independent transition rate

$$w_{BA} = \frac{V}{(2\pi\hbar)^2} \int d^3k \sum_a |\langle B, 1_{\mathbf{k}a}|H_1|A, 0\rangle|^2 \delta(\omega - \omega_B)$$

Use this to find an expression for the lifetime τ.

We remind you of the general relation

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2}[B, [B, A]] + \cdots$$

The Pauli matrices have the form

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They satisfy the relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

and

$$e^{-i\phi\sigma_i} = \cos\phi\, 1 - i\sin\phi\, \sigma_i.$$

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

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This examination paper consists of 4 pages	
Permitted materials:	**Any written material (including internet). You may not collaborate with other students or ask any other person.**

Language: The solutions may be written in Norwegian or English depending on your own preference.
Make sure that your copy of this examination paper is complete before answering.
All answers should be justified

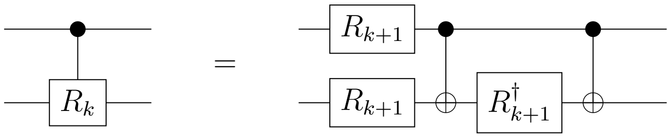
PROBLEM 1

Quantum circuit for controlled R_k

a) In the quantum Fourier transformation, we needed to perform a controlled R_k operation. The one-qubit operator

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

is then performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. We know that all two-qubit operators can be decomposed in single qubit operators and controlled NOT (CNOT) operations. Show that the following quantum circuit is one such decompostion for the controlled R_k operation



b) We consider now general controlled U operations, with U a one-qubit operator. This means that the operation U is performed on the target qubit if the control qubit is in the state $|1\rangle$. When the control qubit is in the state $|0\rangle$ no operation is performed on the target qubit. In both cases, the control qubit is not changed. If this was a classical system, this would be all the possibilities, but in a quantum system, one can have a control qubit that is in a superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ of the two basis states. In general, the two qubits will be entangled by this operation, so no definite quantum state can be ascribed to any of them. However, a special situation arises if the initial state of the target qubit is an eigenstate of U. Draw a quantum circuit desribing this situation. Show that in this case, the two qubits are not entangled by the operation. Show also that in this case, it is the target qubit that is not changed, while the state of the control qubit is changed. Find the final state of the control qubit in terms of the eigenvalues of U.

c) This result is surprising if we only are used to the classical world, and deserves an explanation. Explain in words why the target is not changed while the state of the control does change.

PROBLEM 2
Destruction of entanglement by noise

We have two two-level systems, A and B. Each system has a basis for its Hilbertspace with vector representation

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We introduce a vector representation of the tensor product as described in Problem 5.3 from the exercises. Assume that the density matrix for the joint system is of the form

$$\rho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{1}$$

a) Determine for which values of the parameters a, b, c, d and z this represents a pure state for the joint system.

b) Find the reduced density matrices for systems A and B. In those cases where ρ represents a pure state, determine if the state is entangled or not.

We now specify the Hamiltonian for the two two-level systems as

$$H = \frac{1}{2}\hbar\omega\sigma_z^A + \frac{1}{2}\hbar\omega\sigma_z^B.$$

where $\sigma_z^A = \sigma_z \otimes \mathbb{1}$ and $\sigma_z^B = \mathbb{1} \otimes \sigma_z$. The system is in contact with an environment which means that the density matrix is evolving according to the Lindblad equation

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] - \frac{\gamma}{2}\left[\sigma_+^A\sigma_-^A\rho + \rho\sigma_+^A\sigma_-^A - 2\sigma_-^A\rho\sigma_+^A\right] - \frac{\gamma}{2}\left[\sigma_+^B\sigma_-^B\rho + \rho\sigma_+^B\sigma_-^B - 2\sigma_-^B\rho\sigma_+^B\right].$$

c) What is the temperature of the environment described by this Lindblad equation? Justify your answer.

One can show that if the density matrix at time $t = 0$ is of the form (1) it will have this form at all later times, with time dependent matrix elements $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $z(t)$. If we call the initial values of these variables a_0, b_0, c_0, d_0 and z_0, the solution of the Lindblad equation is

$$\begin{aligned}
a(t) &= a_0 e^{-2\gamma t} \\
b(t) &= b_0 e^{-\gamma t} + a_0 e^{-\gamma t}(1 - e^{-\gamma t}) \\
c(t) &= c_0 e^{-\gamma t} + a_0 e^{-\gamma t}(1 - e^{-\gamma t}) \\
d(t) &= 1 - (b_0 + c_0)e^{-\gamma t} - a_0 e^{-\gamma t}(2 - e^{-\gamma t}) \\
z(t) &= z_0 e^{-\gamma t}
\end{aligned} \tag{2}$$

You do not have to show this, but can use it in the following.

d) Assume the initial conditions

$$a_0 = 1, \qquad b_0 = c_0 = d_0 = z_0 = 0.$$

Find the von Neumann entropy of the state as a function of time. Plot/sketch the entropy as a function of time, and comment on the form of the function.

We have seen that when the full system is in a pure state, we can measure entanglement by the entanglement entropy. If the full system is a mixed state this is not a good measure of entanglement.

e) Give an example of a separable state of two systems where the entropy of entanglement is large.

To study the evolution of the entanglement in our system, we need to quantify the entanglement for the situation where the full system is not in a pure state. One common measure of entanglement is the concurrence. To calculate it we defince the matrix

$$M = \rho\sigma_y^A \otimes \sigma_y^B \rho^* \sigma_y^A \otimes \sigma_y^B$$

where ρ^* is the elementvise complex conjugate of ρ. The concurrence is defined as

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

where λ_i are the square roots of the eigenvalues of M sorted in descending order (λ_1 is the largest, λ_4 is the smallest).

f) Show that the concurrence as a function of time for the density matrix (1) with the elements given by the solution (2) with the initial conditions $d_0 = \frac{1}{3} - a_0$, $b_0 = c_0 = z_0 = \frac{1}{3}$ is

$$C = \max\left(0, \frac{2}{3}e^{-\gamma t} - 2e^{-\gamma t}\sqrt{a_0}\sqrt{1 - \frac{2}{3}e^{-\gamma t} - a_0 e^{-\gamma t}(2 - e^{-\gamma t})}\right).$$

g) Assume now that $a_0 = \frac{1}{3}$. Show that the concurrence goes to 0 in a finite time, and find this time.

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

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	Rottmann: Matematisk formelsamling

Language: The solutions may be written in Norwegian or English depending on your own preference.
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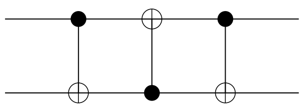
PROBLEM 1
SWAP gate

a) A useful quantum gate is the SWAP gate which interchanges the state of two qubits. That is, if the input state is $|\psi\rangle \otimes |\phi\rangle$ the output will be $SWAP|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Show that the following quantum circuit on two qubits gives a decomposition of the SWAP gate using three CNOT gates.



b) Write the matrix for the SWAP gate. Describe which basis you use.

c) Generalize the above circuit to implement the SWAP gate on two n-qubit registers using only CNOT gates. How many CNOT gates do you need?

PROBLEM 2
Sending information with entangled photons?

The violation of Bell's inequality by certain quantum states is often interpreted as an expression of quantum non-locality. Entangled states are non-local in the sense that one can not assign a pure quantum state to individual particles, but only to the system as a whole. However, this non-locality is of a special form that prevents any information to be transmitted using quantum entanglement. Consider a bipartite system with two parties

A and B. The full system is in a pure quantum state $|\psi\rangle$. Party B could try to transmit information to A in two ways, either making a unitary transformation on subsystem B or a measurement on subsystem B. We will show that neither of these changes the expectation values of observables on subsystem A. We remind you of the Schmidt decomposition of a pure quantum state. For any $|\psi\rangle$ there exist orthonormal bases $|n_i^A\rangle$ and $|n_i^B\rangle$ for the Hilbert spaces of A and B such that

$$|\psi\rangle = \sum_i d_i |n_i^A\rangle \otimes |n_i^B\rangle.$$

a) Define the reduced density matrix of subsystem A and show that the expectation value of all possible observables on subsytem A can be found from the reduced density matrix.

b) Show that the reduced density matrix of A does not change when applying a unitary transformation to B.

c) Show that the reduced density matrix of A does not change when making a measurement on B, as long as we do not know the result of the measurement.

d) What happens with the density matrix for A if we get to know the outcome of the measurement on B?

PROBLEM 3

Charge transfer by adiabatic passage

We have three quantum dots in a row and one electron. Each dot has one state for an electron, so that the electron has three possible states, $|1\rangle$, $|2\rangle$ and $|3\rangle$ (and it can of course also be in superpositions of these). The three basis states are orthogonal and normalized. The motion of the electron can be controlled by gates which change the tunneling amplitude between the dots. The system is described by the Hamiltonian

$$H = -\hbar \begin{pmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & 0 & \Omega_2 \\ 0 & \Omega_2 & 0 \end{pmatrix}.$$

Here Ω_1 is the tunneling amplitude between dots 1 and 2 while Ω_2 is the tunneling amplitude between dots 2 and 3. Both amplitudes are controllable and can be time dependent. The initial state of the electron is $|1\rangle$, which means that the electron is localized on the first dot.

a) Consider first the situation where $\Omega_1 > 0$ is constant and $\Omega_2 = 0$. Find the time dependent state $|\psi(t)\rangle$ if the initial state is $|\psi(0)\rangle = |1\rangle$. Explain in words what this means physically.

b) We now let the matrix elements $\Omega_i(t)$ be time dependent, which means that the Hamiltonian also is time dependent. We define the instantaneous eigenstates and eigenvalues of the Hamiltonian as

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$$

with $n = 1, 2, 3$. Find the states $|n(t)\rangle$ and energies $E_n(t)$ expressed in terms of the matrix elements $\Omega_i(t)$.

c) To study the dynamics of the system we will use a transformed representation of the state. Define the time dependent unitary tranformation $T(t)$ by

$$T(t)|n(0)\rangle = |n(t)\rangle.$$

We define the transformed representation of the state as

$$|\psi'(t)\rangle = T(t)^\dagger|\psi(t)\rangle.$$

Show that the time dependence of the state $|\psi'(t)\rangle$ is given by the Schrödinger equation with a transformed Hamiltonian $H'(t)$ and derive the expression for $H'(t)$ in terms of $H(t)$ and $T(t)$.

d) We now choose the time dependence

$$\Omega_1(t) = \Omega_m e^{-\frac{(t-(t_m+\sigma)/2)^2}{2\sigma^2}}$$
$$\Omega_2(t) = \Omega_m e^{-\frac{(t-(t_m-\sigma)/2)^2}{2\sigma^2}}$$

We want to start the dynamical evolution at $t = 0$ and stop at $t = t_m$. At both these times, we want to have $\Omega_i \approx 0$ so that no tunnelling takes place. This means that we must choose $\sigma \ll t_m$, which implies that

$$\frac{\Omega_1(0)}{\Omega_2(0)} = e^{-t_m/2\sigma} \ll 1.$$

Show that the transformed Hamiltonian $H'(t)$ is

$$H'(t) = -\hbar\Omega(t)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + i\hbar\frac{d\theta}{dt}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{1}$$

where

$$\tan\theta(t) = \frac{\Omega_1(t)}{\Omega_2(t)} \quad \text{and} \quad \Omega(t) = \sqrt{\Omega_1(t)^2 + \Omega_2(t)^2}.$$

e) $\frac{d\theta}{dt}$ can be made arbitrarily small by changing the Hamiltonian slowly. In the following we will assume that the change is so small that we can neglect the final term in (1). We start from $|1\rangle$ at $t = 0$ and evolve slowly in time, find the final state at $t = t_m$.

f) What is the probability of finding the electron in the state $|2\rangle$ during the process? Comment on the result.

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
Day of exam:	**8. December 2022**
Exam hours:	**15.00-19.00, 4 hours**
This examination paper consists of 3 pages	
Permitted materials:	**Approved electronic calculator.**
	Rottmann: Matematisk formelsamling
	One sheet (2 pages) A4 paper of notes

Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before answering.

All answers should be justified

PROBLEM 1

Approximate quantum cloning

The no-cloning theorem tells us that it is impossible to copy an unknown quantum state. In this problem we will study a protocol which takes the quantum state of a two-level system and produces two two-level systems with the state of both approximating as well as possible the original state.

We consider three two-level systems. The first (system A) is the original to be copied, the second (system B) is the system that will receive a copy of the state, and the third (system C) is an auxiliary system (often called an ancilla). We label the states in the usual way, $|000\rangle = |0\rangle_A|0\rangle_B|0\rangle_C = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C$ and similar for the other basis states. We define a unitary operation through the equations

$$U|000\rangle = \sqrt{\frac{2}{3}}|000\rangle + \sqrt{\frac{1}{6}}\left(|011\rangle + |101\rangle\right) \tag{1}$$

$$U|100\rangle = \sqrt{\frac{2}{3}}|111\rangle + \sqrt{\frac{1}{6}}\left(|010\rangle + |100\rangle\right) \tag{2}$$

Let the initial state of system A be $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$, and apply the above operation to the system if the inital state of B and C is $|00\rangle_{BC}$.

a) Calculate the reduced density matrices ρ_A and ρ_B of systems A and B.

b) Determine the Bloch vector of both the final states ρ_A and ρ_B and find how they are related to the Bloch vector of the initial state $|\psi\rangle$.

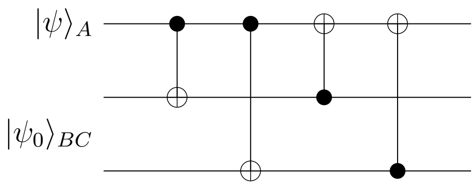
c) We define the fidelity of the copying operation as the overlap of the copied state with the original

$$F = \langle\psi|\rho_B|\psi\rangle.$$

Calculate the fidelity for this operation.

d) The relations (1) and (2) do not define the operation U completely, and it is necessary to show that it can be completed as a unitary operation on all the basis vactors. We can do this by demonstrating that it is produced by a quantum circuit.

Show that the following quantum circuit will implement the unitary operation U on the required input states.



The systems B and C must initially be prepared in the state

$$|\psi_0\rangle_{BC} = \sqrt{\frac{2}{3}}|00\rangle_{BC} + \sqrt{\frac{1}{6}}\left(|01\rangle_{BC} + |11\rangle_{BC}\right).$$

There is a simple circuit to do this step also, starting from the state $|00\rangle_{BC}$, but for the exam we just assume that it has been prepared.

PROBLEM 2

Lindblad equation for pure dephasing

We are interested in studying a two level system subject to pure phase noise. That is, the interaction with the environment does not induce any transitions between the eigenstates of the system. This can be described by a Lindblad equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] - \frac{\gamma}{2}(L^\dagger L\rho + \rho L^\dagger L - 2L\rho L^\dagger)$$

with one Lindblad operator $L = \sigma_z$. The Hamiltonian is

$$H = \frac{1}{2}\hbar\omega_0\sigma_z.$$

a) Solve the Lindblad equation and find the components of the Bloch vector as functions of time for a general initial state. Describe the motion of the Bloch vector.

b) Find an expression for the entropy as a function of time. Sketch the form of the entropy as a function of time and relate the form of the curve to the trajectory of the Bloch vector.

PROBLEM 3
Absolutely maximally entangled states

We start by studying a quantum system that consists of two subsystems, which we cal system A and system B.

a) A product state has a density matrix of the form $\rho = \rho_A \otimes \rho_B$. Show that the entropy of this state is the sum of the individual entropies, $S = S_A + S_B$.

b) We now assume that the Hilbert spaces of the two subsystems have dimensions n_A and n_B. If we have the total system in some pure state $|\psi\rangle$, what is the maximal entanglement entropy that can exist between the two subsystems? You should demonstrate your result, not just state the answer.

Consider the following state of four three-level systems

$$|\psi\rangle = \frac{1}{3} \sum_{i,j=0,1,2} |i\rangle|j\rangle|i+j\rangle|i+2j\rangle$$

where all addtions of the indices i and j are to be taken mod(3). We now select any two of the three-level systems as system A and the remaining two as system B.

c) Calculate the reduced density matrix for all possible divisions of the system i two halves (all possible combinations of two three-level systems in subsystem A) and show that the entanglement entropy is maximal in all cases.

d) From the result of the previous question, what can we say about the entanglement entropy between any of the three-level systems and the remaining three? What can we say about the entanglement between the two three-level systems that constitute subsystem A?