FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2023

Return of solutions:
The problem set is available from Friday morning, 29 September.
You may submit handwritten solutions, but they have to be scanned and included in one single file, which is submitted in Inspera before Friday, 6 October, at 14:00.

Language:

Solutions may be written in Norwegian or English depending on your preference

Questions concerning the problems:

Please ask Joakim Bergli or Maria Markova.

To aid you in solving the problems, you may consult any material that you can find on the topic, but as always you should cite the sources you use.

Problem 1: Quantum error correction

In this problem we will study some basic properties of quantum error correction. To understand the principles, and also the special features of quantum codes it is useful first to get a basic understanding of classical error correction codes. We therefore first consider a classical 3-bit repetition code. Imagine that we are to transmit a message consisting of a string of classical bits. There is a certain (hopefully small) probability p that a bit is changed in transfer and received as the opposite of what it was originally (for simplicity we consider symmetric communication channels, so the probability of a bit flip is the same for both 0 and 1). A simple way to detect and correct such errors is to repeat the same bit, replacing it by a number of identical copies.

a) The simplest classical error correction code is the 3-bit repetition code where each bit is encoded by three identical copies, so that $0 \to 000$ and $1 \to 111$. Explain that using this code one can correct 1 bit-flip error in each triplet. What happens if there are 2 bit-flip errors? What is the probability for this to happen?

In trying to construct similar codes for quantum systems, we face three challenges:

- The no-cloning theorem prevents us from making copies of the original state
- We can not measure the state of the system without collapsing the wavefunction, which means that information encoded in superpositions will be lost
- There are additional errors types possible for a quantum system as compared to a classical.

To illustrate the last point, consider transmitting a qubit which during transfer is changed by the unitary operation

$$U_x(\theta) = e^{i\theta\sigma_x} = \cos\theta I + i\sin\theta\sigma_x \tag{1}$$

where I is the identity operator. if $\theta = \pi/2$ this corresponds to a classical bit flip. Other values of θ are errors with no classical analog.

Nevertheless, it is possible to construct quantum error correction codes, and we will study a simple example. We consider first a 3-qubit repetition code, where each logical state consists of three physical qubits

$$|0\rangle_b = |000\rangle \qquad |1\rangle_b = |111\rangle.$$

The subscipt b on the states $|\cdot\rangle_b$ expresses the fact that this code will correct bit-flip errors. When encoding a general state

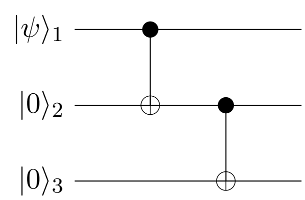
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

we prepare the three qubits in the state

$$|\psi\rangle_b = \alpha|000\rangle + \beta|111\rangle.$$

b) Explain why this encoding process does not violate the no-cloning theorem.

c) Show that the following quantum circuit will implement the encoding if the original state $|\psi\rangle_1$ initially is stored in qubit 1.

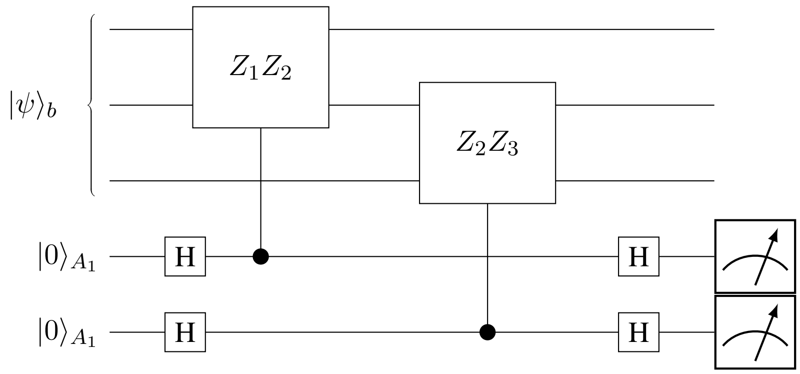


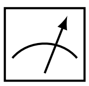
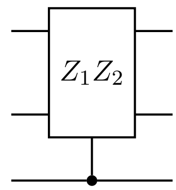
We study first errors that correspond to classical bit-flip errors. These are described by the unitary operators X_i, flipping qubit i. That is, $X_i = \sigma_x^{(i)}$, the Pauli matrix σ_x acting on qubit i. Similarly, we write $Y_i = \sigma_y^{(i)}$ and $Z_i = \sigma_z^{(i)}$. The two operators Z_1Z_2 and Z_2Z_3 are called the stabilizers of the code, and the two logical states $|0\rangle_b$ and $|1\rangle_b$ are eigenstates of both stabilizers with eigenvalue 1.

d) Consider all states that are reached from the two logical states by a single bit flip on one of the qubits. Check that they are all eigenstates of the stabilizers and find the corresponding eigenvalues. Explain how we can use the measurement of the stabilizers to determine an operation that will correct the error. Explain why this will still work if the initial state is a superposition of the two logical states.

e) Explain the difference between measuring the stabilizer Z_1Z_2 and measuring the operators Z_1 and Z_2 separately. What information about the state do we get in the two cases?

f) Measuring the stabilizers directly is not always possible, as it involves a joint measurement on two qubits. Instead, one can transfer the information to ancillas (additional qubits) and measure their state instead. Show that the following circuit is equivalent to measuring the two stabilizers.



 indicates that the qubit is measured in the $\{|0\rangle, |1\rangle\}$-basis and 

means that the operation Z_1Z_2 is executed on the first two qubits if the third qubit is in state $|1\rangle$ while no operation is done if the third qubit is in state $|0\rangle$ (controlled Z_1Z_2 operation).

So far, we considered only spin-flip errors on the individual qubits, but we know that in contrast to classical bits, qubits can have arbitrary rotations of the state, as described by the unitary operation $U_x(\theta)$ given in Eq. (1).

g) Study the what happens when measuring the stabilizers after such an error has occurred. Explain why the same code as before also can correct errors of this type.

Qubits can also experience phase errors which do not have any classical analog. These are the effect of the operators

$$U_z(\phi) = e^{i\phi\sigma_z} = \cos\phi I + i\sin\phi\sigma_z$$

acting on the qubit.

h) Investigate what happens if a phase error affects one of the qubits in the 3-qubit bit-flip repetition code that we have studied. Show that this code will not be able to handle phase errors.

We can construct a code that detects and corrects phase errors in the same way as the 3-qubit bit flip code corrects bit flip errors. This is not difficult once we realize that the operator Z is acting like a bit flip between the eigenstates of X, so we can use the X eigenstates as the basis states in the same way as we used the Z eigenstates for the bit flip code. We denote the X eigenstates as

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

In the 3-qubit phase flip code the logical states are then

$$|0\rangle_p = |+++\rangle \qquad |1\rangle_p = |---\rangle$$

and the stabilizers are X_1X_2 and X_2X_3.

Protecting against both bit flip and phase flip errors at the same time is then achieved by what is called concatenation of the two codes. Three logical qubits in the bit flip code are then used in the phase flip code to get a 9-qubit code with the logical states

$$|0\rangle_L = |+++\rangle_b \qquad |1\rangle_L = |---\rangle_b$$

where

$$|+\rangle_b = \frac{1}{\sqrt{2}}(|0\rangle_b + |1\rangle_b), \qquad |-\rangle_b = \frac{1}{\sqrt{2}}(|0\rangle_b - |1\rangle_b).$$

i) Explain why the stabilizers are the operators Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

j) Show that all the stabilizers of the 9-qubit code commute so that they can be simultaneously measured.

k) Show that any error (unitary operation) on a single qubit can be written as

$$U = a_0 I + a_1 X + a_2 XZ + a_3 Z$$

with complex coefficients a_i. This means that the 9-qubit code that corrects bit-flip and phase-flip errors will correct any 1-qubit error.

Problem 2: Encoding a qbit in an oscillator

In this second problem we will derive some properties of a different type of quantum error correcting code: the Gottesman-Kitaev-Preskill code. This code was one of the first propositions for a so-called bosonic code: a scheme in which the logical qubit - a finite-dimensional quantum system - is encoded into an infinite-dimensional system such as a harmonic oscillator.

a) We first define the displacement operator $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators respectively such that $[\hat{a}, \hat{a}^\dagger] = 1$, and α is a complex number. Show that $\hat{D}(\beta)\hat{D}(\alpha) = e^{(\beta\alpha^* - \beta^*\alpha)/2}\hat{D}(\alpha+\beta) = e^{\beta\alpha^* - \beta^*\alpha}\hat{D}(\alpha)\hat{D}(\beta)$ (show both equalities).

The GKP code is designed to protect against small displacement errors. For this purpose, it encodes a qubit's states into a lattice of regular displacements from the origin in the phase space of a harmonic oscillator. This means that the if we denote the eigenstates of the position operator $\hat{x} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$ by $|x\rangle$, the logical states are

$$|0\rangle_L = \sum_{j=-\infty}^{\infty} |2j\alpha\rangle,$$

$$|1\rangle_L = \sum_{j=-\infty}^{\infty} |(2j+1)\alpha\rangle$$

b) Show that for real α we have that

$$\hat{D}(\alpha)|x\rangle = |x+\alpha\rangle.$$

c) Show that for real α we have

$$\hat{D}(\alpha)|0\rangle_L = |1\rangle_L, \qquad \hat{D}(\alpha)|1\rangle_L = |0\rangle_L.$$

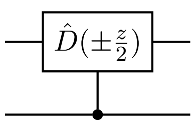
The above result means that the operator $\bar{X} = \hat{D}(\alpha)$ acts on the logical states in the same way as the usual Pauli operator σ_x. We define the logical Pauli operators $\bar{X} = \hat{D}(\alpha)$, $\bar{Z} = \hat{D}(\beta)$, and $\bar{Y} = i\bar{X}\bar{Z}$.

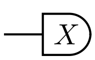
d) Check that if $\beta\alpha^* - \beta^*\alpha = i\pi$, these operators have the same commutation relations as the usual Pauli matrices. In the following we are going to assume that α is real and that β is purely imaginary.

e) If we take the logical states is the position representation, the wavefunctions consist of a set of equally spaced δ-functions, often called a Dirac comb. Show that if we instead use the momentum representation wavefunctions, they also have the same structure, with δ-functions separated by the distance 2β.

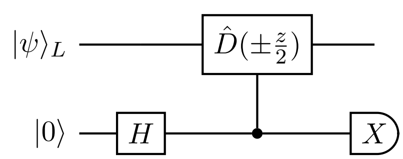
It may be useful to know that

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-2\pi n i t/T}.$$

f) Phase estimation circuit: In the following circuit, the upper line represents the harmonic oscillator while the lower line is a qubit (two level system). The controlled displacement gate $C\hat{D}(\pm\frac{z}{2})$ represented by 

applies a displacement $\hat{D}(\pm\frac{z}{2})$ on the target oscillator depending on the value of the control qubit: $C\hat{D}(\pm\frac{z}{2}) = \hat{D}(z/2)\otimes|0_a\rangle\langle 0_a| + \hat{D}(-z/2)\otimes|1_a\rangle\langle 1_a|$, and 

represents a measurement in the X basis.



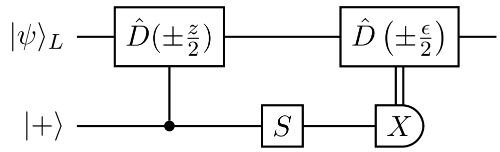
(2)

Show that the measurement outcome probabilities for this circuit are:

$$P(\pm) = \frac{1}{2}\left[1 \pm \frac{1}{2}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle + \langle\psi_L|\hat{D}^\dagger(z)|\psi_L\rangle\right)\right] \tag{3}$$

.

The following circuit is the so-called "Sharpen" circuit used for the preparation of GKP states.



(4)

$S = \text{diag}(1, i)$ is the phase-shift gate. ϵ is a small parameter with $\arg(\epsilon) = \arg(z) - \pi/2$ such that $\hat{D}(\epsilon)$ is a small displacement in a direction orthogonal to $\hat{D}(z)$. The double line from the X-measurement symbol means that the operation $\hat{D}\left(\pm\frac{\epsilon}{2}\right)$ is made with the $\pm$-sign depending on the outcome of the measurement.

g) Show that the probabiliy of getting $\pm$ outcome on the ancilla qubit is $P(\pm) = \frac{1}{2}\left(1 \pm \text{Im}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle\right)\right)$. Show that if the input state $|\psi\rangle_L$ is an eigenstate of $\hat{D}(z)$ with eigenvalue $e^{i\theta}$, then for a $\pm$ outcome, the output is an eigenstate with eigenvalue $e^{i(\theta \mp z\epsilon)}$. Explain how repeated use of this circuit will result in the state approaching an eigenstate with eigenvalue 1.