FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2023. Solution

Problem 1: Quantum error correction

a) If there is one bit flip we have that 000 is received as 100, 010, or 001. By majority vote we will correctly change this to 000, correcting the error. If two bit flips would happen, 000 could be received as 110, which we would erroneously correct to 111. The probability for this to happen is p^2. If p
1 we ahve that $p^2 \ll p$, so the probability that we get the correct result after correction is much larger that without error correction.

b) Cloning the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ would give the state $|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \neq \alpha|000\rangle + \beta|111\rangle$. That the given encoding operation is in fact unitary is proven by the fact that it is the result of the circuit in the next question.

c) If the initial state $|\psi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1$ is initially stored in the first qubit we have that the action of the circuit is

$$
\begin{aligned}
|\psi\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 = \alpha|000\rangle + \beta|100\rangle& \\
\overset{CNOT_{12}}{\rightarrow} \alpha|000\rangle + \beta|110\rangle& \\
\overset{CNOT_{23}}{\rightarrow} \alpha|000\rangle + \beta|111\rangle&.
\end{aligned}
$$

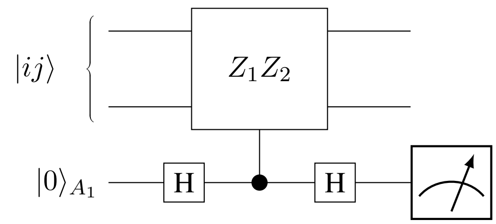
d) We get the following table of eigenvalues of Z_1Z_2 and Z_2Z_3 for the possible cases

Initial state	Error	State	Z_1Z_2	Z_2Z_3
$\|000\rangle$	X_1	$\|100\rangle$	-1	1
	X_2	$\|010\rangle$	-1	-1
	X_3	$\|001\rangle$	1	-1
$\|111\rangle$	X_1	$\|011\rangle$	-1	1
	X_2	$\|101\rangle$	-1	-1
	X_3	$\|110\rangle$	1	-1

As we see, all states are eigenstates of both the stabilizers. The measurement of both stabilizers will then give the corresponding eigenvalues as results. We see that these two eigenvalues uniquely determine which error has happened, independent of the initial state. This also means that the same will be true if we start from a superposition of the two logical states. To correct the error, we then have to apply the corresponding inverse operation (which is the same as the original error since $X_i^2 = I$, the identity). For example, if the measurement of Z_1Z_2 gives 1 and Z_2Z_3 gives 1 we know that the error was X_1 and correct it by applying the operation X_1 to the first qubit. Note that the measurement of the stabilizers only gives information about the error, and no information at all about the initial state. Otherwise we would affect the information stored in the initial state.

e) Measuring the operators Z_1 and Z_2 separately will tell us the state of each qubit individually. This would also reveal information about the initial state and collapse the wavefunction in a way that perturbs the information. Measuring $Z_1 Z_2$ will only tell if the two qubits are in the same state (when the result is 1) or opposite states (when the result is -1). This does not reveal any information about the initial state.

f) It is sufficient to consider only one of the stabilizers. We therefore analyze the simpler circuit



which only involves the first two qubits and the first ancilla. We also only need to consider the action of the circuit on the basis states $|ij\rangle$, and we find that

$$\begin{aligned}
|ij0\rangle &\overset{H}{\to} \frac{1}{\sqrt{2}}(|ij0\rangle + |ij1\rangle) \\
&\overset{CZ_1Z_2}{\to} \frac{1}{\sqrt{2}}(|ij0\rangle + Z_1 Z_2 |ij1\rangle) \\
&\overset{H}{\to} \frac{1}{2}(|ij0\rangle + |ij1\rangle + Z_1 Z_2 |ij0\rangle - Z_1 Z_2 |ij1\rangle) \\
&= \frac{1}{2}(1 + Z_1 Z_2)|ij0\rangle + \frac{1}{2}(1 - Z_1 Z_2)|ij1\rangle \\
&= \begin{cases} |ij0\rangle & \text{if } i = j \\ |ij1\rangle & \text{if } i \neq j \end{cases}
\end{aligned}$$

So it means that measuring the ancilla A_1 will give 0 if $Z_1 Z_2 |ij\rangle = 1$ and 1 if $Z_1 Z_2 |ij\rangle = -1$.

g) Consider for example starting from the state $|000\rangle$ and having such an error act on the first qubit

$$|000\rangle \to U_x^{(1)}(\theta) = \cos\theta |000\rangle + i \sin\theta |100\rangle.$$

Measuring $Z_1 Z_2$ will give $+1$ with the probability $\cos^2\theta$ with the state collapsing to $|000\rangle$ and -1 with the probability $\sin^2\theta$ with the state collapsing to $|100\rangle$. In either case, there is correspondence between the outcome of the measurement and the resulting state (as always). In a sense we can say that the measurement decides if the first bit was unchanged or that the bit flip occured. After the measurement, we can correct the error as before if it happened, or do nothing if no error was detected.

h) Consider for definiteness that the error acts on the first qubit and that $\phi = \pi/2$. If the initial state is $|\phi\rangle = \alpha|000\rangle + \beta|111\rangle$ the state after the error occurs is

$$|\phi'\rangle = U_z^{(1)}(\pi/2)|\phi\rangle = \alpha|000\rangle - \beta|111\rangle.$$

We have that

$$Z_1 Z_2|\phi'\rangle = Z_2 Z_3|\phi'\rangle = |\phi'\rangle$$

which means that the error is not detected.

i) The stabilizers for the correction of bit flips on the individual qubits are Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8 and Z_8Z_9. For the phase flip we have to use the stabilizers on the qbits that are constructed of bit-flip corrected triples of qubits. These are $X_1^bX_2^b$ and $X_2^bX_3^b$, where $X_1^b = X_1X_2X_3$, $X_2^b = X_4X_5X_6$ and $X_3^b = X_7X_8X_9$, which means that the stabilizers are $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

j) All the Z_iZ_j commute and also $[X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9] = 0$. We only have to check commutators of the type

$$[Z_1Z_2, X_1X_2X_3X_4X_5X_6] = [Z_1Z_2, X_1X_2]X_3X_4X_5X_6$$

as operators on different qubits commute. We have

$$[Z_1Z_2, X_1X_2] = Z_1[Z_2, X_1X_2]+[Z_1, X_1X_2]Z_2 = Z_1X_1[Z_2, X_2]+[Z_1, X_1]X_2Z_2 = iY_12iY_2+2iY_1(-iY_2) = 0.$$

k) Any unitary operation is of the form $U = e^{-iHt}$ for a suitable Hamiltonian H and time t (we use units where $\hbar = 1$). The Hamiltonian, being Hermitian, can be expanded in the Pauli matrices and the identity I as $H = h_0I + \sum_i h_i\sigma_i = h_0I + h\sigma_{\mathbf{n}}$ where the unit vector $\mathbf{n} = \mathbf{h}/|\mathbf{h}|$, $h = |\mathbf{h}|$ and $\sigma_{\mathbf{n}} = \mathbf{h}\cdot\sigma$. We know that these operators have the property $\sigma_{\mathbf{n}}^2 = I$, which means that

$$U = e^{-ih_0t}(\cos htI - i\sin ht\sigma_{\mathbf{n}} = a_0I + a_1X + a_2XZ + a_3Z$$

since we have that $Y = -iXZ$.

Problem 2: Encoding a qbit in an oscillator

a) We use the BCH formula

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+cdots}$$

where the higher order terms in the exponent on the right involve commutators of $[A, B]$ with other operators. Defining $E_\alpha = \alpha\hat{a}^\dagger - \alpha^*\hat{a}$ we have that the commutator

$$[E_\beta, E_\alpha] = \beta\alpha^* - \beta^*\alpha$$

is a number, and therefore commutes with all operators and the series terminates. We then get

$$\hat{D}(\beta)\hat{D}(\alpha) = e^{E_\beta}e^{E_\alpha} = e^{E_\beta+E_\alpha+\frac{1}{2}[E_\beta,E_\alpha]} = e^{E_\beta+E_\alpha+\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)} = \hat{D}(\alpha+\beta)e^{\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)}.$$

Similarly we find that

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)e^{-\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)}.$$

This gives

$$\hat{D}(\beta)\hat{D}(\alpha) = e^{\beta\alpha^*-\beta^*\alpha}.$$

b) We use the expansion

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2}[B, [B, A]] + cdots$$

to get

$$\hat{D}(\alpha)^\dagger \hat{x} \hat{D}(\alpha) = e^{-E_\alpha} \hat{x} e^{E_\alpha} = \hat{x} + [\hat{x}, E_\alpha] = \hat{x} + \alpha.$$

Then we check that $\hat{D}(\alpha)|x\rangle$ is an eigenstate of $\hat{x}$

$$\hat{x}\hat{D}(\alpha)|x\rangle = \hat{D}(\alpha)\hat{D}(\alpha)^\dagger \hat{x}\hat{D}(\alpha)|x\rangle = \hat{D}(\alpha)(\hat{x}+\alpha)|x\rangle = (x+\alpha)\hat{D}(\alpha)|x\rangle$$

which shows that $\hat{D}(\alpha)|x\rangle = |x+\alpha\rangle$.

c)

$$\hat{D}(\alpha)|0\rangle_L = \sum_{j=-\infty}^{\infty} \hat{D}(\alpha)|2j\alpha\rangle = \sum_{j=-\infty}^{\infty} |2j\alpha + \alpha\rangle = |1\rangle_L,$$

and similarly to show $\hat{D}(\alpha)|1\rangle_L = |0\rangle_L$.

d) We must have $\bar{X}\bar{Z} = -\bar{Z}\bar{X}$ which gives

$$\hat{D}(\alpha)\hat{D}(\beta) = -\hat{D}(\beta)\hat{D}(\alpha) = e^{\beta\alpha^* - \beta^*\alpha}\hat{D}(\alpha)\hat{D}(\beta).$$

For this to be true we must have $\beta\alpha^* - \beta^*\alpha = i\pi$. Assuming from now on that α is real and β purely imaginary we can then write this as $\beta = i\frac{\pi}{2\alpha}$ and get

$$\hat{D}(\beta)|0\rangle_L = e^{i\frac{\pi}{2\alpha}(\hat{a}^\dagger + \hat{a})}|0\rangle_L = e^{i\frac{\pi}{\alpha}\hat{x}}|0\rangle_L = e^{i\frac{\pi}{\alpha}\hat{x}} \sum_{j=-\infty}^{\infty} |2j\alpha\rangle = \sum_{j=-\infty}^{\infty} e^{i\frac{\pi}{\alpha}2\alpha j}|2j\alpha\rangle = \sum_{j=-\infty}^{\infty} e^{i2\pi j}|2j\alpha\rangle = |0\rangle_L$$

Similarly we get that $\hat{D}(\beta)|1\rangle_L = -|1\rangle_L$, so $\bar{Z} = \hat{D}(\beta)$ acts in the desired way on the logical states. Using this we then easily check the commutation relations.

e) In the position representation we have the wavefuction

$$\psi_0(x) = \langle x|0\rangle_L = \sum_{j=-\infty}^{\infty} \delta(x - 2\alpha j) = \frac{1}{2\alpha}\sum_{j=-\infty}^{\infty} e^{-\frac{i\pi}{\alpha}jx}$$

In the momentum repersentation we have

$$\bar{\psi}_0(p) = \int dx e^{ipx}\phi_0(x) = \frac{1}{2\alpha}\sum_{j=-\infty}^{\infty} \int e^{ipx} e^{-\frac{i\pi}{\alpha}jx} = \frac{\pi}{\alpha}\sum_{j=-\infty}^{\infty} \delta(p - 2|\beta|j).$$

For the state $|1\rangle_L$ we get

$$\bar{\psi}_1(p) = \frac{\pi}{\alpha}\sum_{j=-\infty}^{\infty} (-1)^j \delta(p - 2|\beta|j).$$

f) The action of the circuit on an initial state $|\psi\rangle_L$ is

$$
\begin{aligned}
|\psi_L\rangle \otimes |0\rangle &\xrightarrow{H} \frac{1}{\sqrt{2}}|\psi_L\rangle \otimes (|0\rangle + |1\rangle) \\
&\overset{C\hat{D}(\pm\frac{z}{2})}{\rightarrow} \frac{1}{\sqrt{2}}\left[\hat{D}(\frac{z}{2})|\psi_L\rangle \otimes |0\rangle + \hat{D}(-\frac{z}{2})|\psi_L\rangle \otimes |1\rangle\right] \\
&= \frac{1}{2}\left\{\left[\hat{D}(\frac{z}{2}) + \hat{D}(-\frac{z}{2})\right]|\psi\rangle_L \otimes |+\rangle + \left[\hat{D}(\frac{z}{2}) - \hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \otimes |-\rangle\right\},
\end{aligned}
$$

which gives

$$
\begin{aligned}
P_\pm &= \frac{1}{4}\langle\psi_L|\left[\hat{D}^\dagger(\frac{z}{2}) \pm \hat{D}^\dagger(-\frac{z}{2})\right]\left[\hat{D}(\frac{z}{2}) \pm \hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= \frac{1}{4}\left[2 \pm \langle\psi_L|\hat{D}^\dagger(z) + \hat{D}(z)|\psi_L\rangle\right] \\
&= \frac{1}{2}\left[1 \pm \frac{1}{2}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle + \langle\psi_L|\hat{D}^\dagger(z)|\psi_L\rangle\right)\right].
\end{aligned}
$$

g) The circuit does the following

$$
\begin{aligned}
|\psi_L\rangle \otimes |0\rangle &\xrightarrow{H} \frac{1}{\sqrt{2}}|\psi_L\rangle \otimes (|0\rangle + |1\rangle) \\
&\overset{C\hat{D}(\pm\frac{z}{2})}{\rightarrow} \frac{1}{\sqrt{2}}\left[\hat{D}(\frac{z}{2})|\psi_L\rangle \otimes |0\rangle + \hat{D}(-\frac{z}{2})|\psi_L\rangle \otimes |1\rangle\right] \\
&\xrightarrow{S} \frac{1}{\sqrt{2}}\left[\hat{D}(\frac{z}{2})|\psi_L\rangle \otimes |0\rangle + i\hat{D}(-\frac{z}{2})|\psi_L\rangle \otimes |1\rangle\right] \\
&= \frac{1}{2}\left\{\left[\hat{D}(\frac{z}{2}) + i\hat{D}(-\frac{z}{2})\right]|\psi\rangle_L \otimes |+\rangle + \left[\hat{D}(\frac{z}{2}) - i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \otimes |-\rangle\right\}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
P_\pm &= \frac{1}{4}\langle\psi_L|\left[\hat{D}^\dagger(\frac{z}{2}) \mp i\hat{D}^\dagger(-\frac{z}{2})\right]\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= \frac{1}{2}\left(1 \pm \mathrm{Im}\left(\langle\psi_L|\hat{D}(z)|\psi_L\rangle\right)\right).
\end{aligned}
$$

If

$$
\hat{D}(z)|\psi_L\rangle = e^{i\theta}|\psi_L\rangle
$$

the state after measurement outcome $\pm$ is

$$
|\psi'_L\rangle = \hat{D}(\pm\frac{\epsilon}{2})\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle.
$$

We check that this is and eigenstate of $\hat{D}(z)$

$$
\begin{aligned}
\hat{D}(z)|\psi'_L\rangle &= \hat{D}(z)\hat{D}(\pm\frac{\epsilon}{2})\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= \hat{D}(\pm\frac{\epsilon}{2})\hat{D}(z)e^{\mp\frac{1}{2}(z\epsilon^*-z^*\epsilon)}\left[\hat{D}(\frac{z}{2}) \pm i\hat{D}(-\frac{z}{2})\right]|\psi_L\rangle \\
&= e^{i\theta\mp\frac{1}{2}(z\epsilon^*-z^*\epsilon)}|\psi'_L\rangle \\
&= e^{i(\theta\mp|z\epsilon|)}|\psi'_L\rangle.
\end{aligned}
$$

We also have that

$$
P_\pm = \frac{1}{2}(1 \pm \sin\theta)
$$

so that if $\theta > 0$ the probability of measuring $+$ is greater thatn $\frac{1}{2}$. In that case, we se that the new exponent $\theta \mp |z\epsilon| < \theta$, so the angle θ is on average reduced. On the other hand, if $\theta < 0$ it is more likely to measure $-$, and the angle will increase. So the state will approach the state with $\theta = 0$, that is, the eigenvalue will be 1.