Problem set 10

10.1 Gaussian integrals

The following formula gives the integral of a gaussian function

$$I \equiv \int_{-\infty}^{\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} \tag{1}$$

This is correct for complex λ provided the real part of λ is positive. Verify this by evaluating the square I^2 as a two-dimensional integral

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dy\, e^{-\lambda(x^2+y^2)} \tag{2}$$

and by changing to to polar coordinates in the evaluation.

Determine also the integral

$$I' \equiv \int_{-\infty}^{\infty} dx\, e^{-\lambda x^2 + ax + b} \tag{3}$$

with two additional parameters, a and b.

10.2 Path integral for free particle

We will make a direct calculation of the propagator for a free particle. Start from the discretized path integral, Eq (1.101) in the lecture notes, with the potential term $V(x) = 0$. We are going to calculate each of the integrals succesively.

a) Show first that for the terms containing x_1 we have

$$I_1 = N_{\Delta t}^2 \int dx_1 e^{\frac{im}{2\hbar\Delta t}\left[(x_1 - x_i)^2 + (x_2 - x_1)^2\right]} = \sqrt{\frac{m}{2\pi i\hbar \cdot 2\Delta t}} e^{\frac{im}{2\hbar \cdot 2\Delta t}(x_2 - x_i)^2}$$

b) Multiply by the remaining term containing x_2 and show that

$$I_2 = N_{\Delta t} \int dx_2 e^{\frac{im}{2\hbar\Delta t}(x_3 - x_2)^2} I_1 = N_{\Delta t} \sqrt{\frac{m}{2\pi i\hbar \cdot 2\Delta t}} \int dx_2 e^{\frac{im}{2\hbar\Delta t}\left[\frac{1}{2}(x_2 - x_i)^2 + (x_3 - x_2)^2\right]}$$
$$= \sqrt{\frac{m}{2\pi i\hbar \cdot 3\Delta t}} e^{\frac{im}{2\hbar \cdot 3\Delta t}(x_3 - x_i)^2}$$

Notice how this is similar to the previous step, only with 3 replacing 2 in several places. This pattern will continue for the following steps.

c) Using this pattern prove/guess the final result after all $n-1$ integrals and compare to the result (1.109) in the lecture notes.

10.3 Path integral for harmonic oscillator

We will calculate the propagator for a harmonic oscillator by evaluating the path integral using the same method as in was done for the free particle in Eqs (1.105)-(1.108) in the lecture notes.

a) Using the Fourier expansion (1.105) in the harmonic oscillator Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$, show that the action can be written

$$S[x(t)] = S[x_{cl}(t)] + \frac{mT}{4}\sum_n \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right] c_n^2$$

b) Evaluate the path integral in the form of integrals over the Fourier coefficients c_n as in Eq (1.107) in the lecture notes and show that the propagator is

$$\mathcal{G}(x_f t_f, x_i t_i) = N e^{\frac{i}{\hbar}S[x_{cl}(t)]} \prod_n \left[1 - \left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}$$

Where N is a ω-independent normalization factor. To determine the normalization we can take the limit $\omega \to 0$ and compare to the result for a free particle that we found in Problem 2.2. You will also need the product formula

$$\prod_n \left(1 - \frac{a^2}{n^2}\right) = \frac{\sin a\pi}{a\pi}$$

In the end you should find that

$$\mathcal{G}(x_f t_f, x_i t_i) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}}\, e^{\frac{i}{\hbar}S[x_{cl}(t)]}$$

where $T = t_f - t_i$.

c) We still need the action along the classical path, prove that

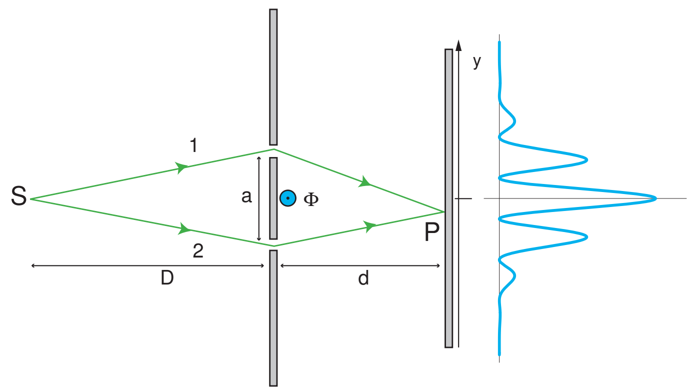
$$S[x_{cl}(t)] = \frac{m\omega}{2\sin\omega T}\left[(x_f^2 + x_i^2)\cos\omega T - 2x_f x_i\right]$$

Warning: Even if this is a simple problem in classical mechanics, the calculations may be a bit long.

d) Use the classical action from the previous question in Eq (1.119) of the lecture notes and find the semiclasical propagator (1.116) for the harmonic oscillator. Compare it to the exact solution found above, and confirm that the semiclassical approximation is exact in this case, as expected since the Lagrangian is quadratic.

10.4 The Aharonov-Bohm effect

We consider a double slit experiment as sketched in the figure. Electrons are emitted from a source S and can pass trough one of the two slits of a first screen before being registered on a second screen.



When a large number of electrons are registered they are found to form an interference patterns with minima and maxima on the screen.

Behind the middle part of the first screen a solenoid is placed which carries a magnetic flux Φ. The direction of the solenoid is parallel to the direction of the two slits, so that the paths through the upper slit pass on one side of the solenoid and the paths through the lower slit pass on the other side of the solenoid. We consider the magnetic field to be completely screened from the region where the electrons move, so that at no point along the trajectories of the electron there is a magnetic force acting on the particles. Nevertheless, quantum theory predicts that the strength of the magnetic flux will influence the interference pattern so that the maxima and minima are shifted up or down when the flux is changed. This is called the Aharonov-Bohm effect.

We consider in the following the distance d between the screens and the distance D between the source and the first screen to be much larger than the distance a between the two slits, and also to be much larger than the distance y from the central point of the second screen to any point P where an electron is registered.

As a reminder the classical Lagrangian of an electron moving in a magnetic field is

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}}^2 + e\mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}} \tag{4}$$

with $\mathbf{A}$ as the vector potential, and the magnetic field thus given as $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. As follows from Stokes' theorem the magnetic flux is given as the line integral

$$\Phi = \oint_C \mathbf{A} \cdot d\mathbf{r} \tag{5}$$

where C is any given closed loop that encircles once the solenoid.

a) Show that the vector potential

$$A(\mathbf{r}) = \frac{k}{r^2}(-y, x, 0) \tag{6}$$

with $r = \sqrt{x^2 + y^2}$ provides a magnetic field concentrated in an infinitesimal solenoid at the origin, with no field outside of this in agreement with the above assumptions. Find the value of k expressed trough the magnetic flux through the soleniod.

We consider the situation where a single electron is emitted at time $t = 0$ from the source and is registered at a later time t at a point P of the screen. The probability distribution over the screen for where the electron is registered can be written as

$$p(y) = \lambda |\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0)|^2 \tag{7}$$

with y as the vertical coordinate of P, λ as a proportionality factor and $\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0)$ as the propagator from the initial point $(\mathbf{r}_S, 0)$ to the final point $(\mathbf{r}_P, t)$.

We consider in the following the the semi-classical approximation to the propagator, which we write as

$$\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0) = N \sum_{n=1}^{2} e^{\frac{i}{\hbar} S_n} \tag{8}$$

where S_n is the action integral for classical free-particle motion either through the upper slit ($n = 1$) or through the lower slit ($n = 2$), and N is a (y-dependent) normalization factor which is assumed to be independent of the path. Since the classical motion is not affected by the magnetic field, both λ and N are independent of the magnetic flux.

b) Show that the probability $p(y)$ depends on the *difference* between the action integrals of the two paths.

c) Show that the difference between the two action integrals can be written as a function of the magnetic flux Φ.

d) In classical electrodynamics, it is often stated that the real, pysical fields are the electric and magnetic fields, while the scalar and vector potentials are auxiliary quantities which do not have any direct physical counterpart. This is argued from the fact that one can shift the potentials by the gauge transformation

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\chi, \qquad \phi \to \phi' = \phi + \frac{\partial \chi}{\partial t} \tag{9}$$

where χ is an arbitrary function of space and time (if you are not familiar with gauge transformations, see the lecture notes, section 4.4.1). What are the implications of the Aharonov-Bohm effect on this picture? How can the electron be affected by the presence of a magnetic field in a region where the electron wavefunction is zero?

e) Does the Aharonov-Bohm effect depend on the choice of gauge?

f) Show that the probability $p(y)$ depends periodically on the magnetic flux Φ. What is the flux period? Describe qualitatively how the interference pattern changes with variations in Φ.