Problem set 11

11.1 The canonical commutation relations.

The basic commutation relations of the electromagnetic field can be written as

$$\left[\hat{A}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right] = -i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}, \qquad \left[\hat{A}_{\mathbf{k}a}, \hat{A}^\dagger_{\mathbf{k}'a'}\right] = \left[\hat{E}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right] = 0, \tag{1}$$

where $\mathbf{k}$ is the wave vector of the electromagnetic field and a is a polarization index. The photon creation and annihilation operators are related to the field operators by

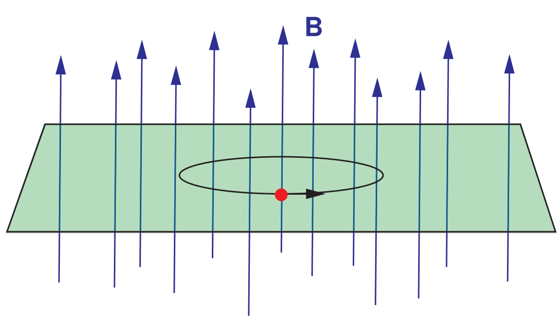
$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}(\hat{a}_{\mathbf{k}a} + \hat{a}^\dagger_{-\mathbf{k}\bar{a}})\,, \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}(\hat{a}_{\mathbf{k}a} - \hat{a}^\dagger_{-\mathbf{k}\bar{a}}) \tag{2}$$

where $\bar{a}$ is related to a by a permutation of the two values of the polarization index. Use the above expressions to show that the creation and annihilation operators satisfy the standard harmonic oscillator commutation relations,

$$\left[\hat{a}_{\mathbf{k}a}, \hat{a}^\dagger_{\mathbf{k}'a'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{3}$$

11.2 Charged particle in a strong magnetic field (Midterm Exam 2005).

We study in this problem a particle with electric charge e that moves in a strong magnetic field $\mathbf{B}$. The motion is assumed to be constrained to a plane (the x, y-plane) with the magnetic field orthogonal to the plane. The magnetic field is assumed to be constant over the plane, and eB is taken to be *negative*, with B as the z-component of $\mathbf{B}$. We assume in the following that the rotationally symmetric form of the vector potential is chosen, $\mathbf{A} = -(1/2)\mathbf{r} \times \mathbf{B}$. The relation between velocity and (canonical) momentum is $\mathbf{v} = (\mathbf{p} - e\mathbf{A})/m$, and the Hamiltonian has the standard form $H = (1/2m)(\mathbf{p} - e\mathbf{A})^2$.



We consider first the classical, non-relativistic form of the particle motion. Next we study the quantum description, where a set of coherent states is introduced for the particle in the degenerate ground state of the Hamiltonian. This description is particularly relevant for the study of the quantum Hall effect, where a 2-dimensional electron gas moves under the influence of a strong magnetic field.

a) Use Newton's second law for a charged particle in a magnetic field to show that, classically, the particle moves in a circular orbit with constant angular velocity $\omega = -eB/m$. Show, by use of the equation of motion, that generally the (z component of the) mechanical angular momentum $L_{mek} = m(xv_y - yv_x)$ is *not* a constant of motion, whereas $L = L_{mek} + (eB/2)r^2$ is conserved. (The last term can be viewed as an electromagnetic contribution.)

b) Consider the following vector, $\mathbf{R} = \mathbf{r} + (1/\omega)\mathbf{k}\times\mathbf{v}$, with $\mathbf{r}$ as the position and $\mathbf{v}$ as the velocity, $\mathbf{k}$ as the unit vector in the z-direction (orthogonal to the plane) and B as the (z-component of) the magnetic field. Show that $\mathbf{R}$ is a constant of motion, and give a physical interpretation of $\mathbf{R}$ and of the vector $\boldsymbol{\rho} = (1/\omega)\mathbf{k}\times\mathbf{v}$ for the circular orbits? $\mathbf{R}$ is known as the guiding center coordinate.

c) In the quantum description, the position $\mathbf{r}$ and momentum $\mathbf{p}$ are, in the standard way replaced by operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ that satisfy the Heisenberg commutation relations. Show that the two components $\hat{X}$ and $\hat{Y}$ of the vector $\hat{\mathbf{R}}$, in the quantized form, do not commute. In what sense is the X, Y-plane similar to a two-dimensional phase space? Examine also commutators between the components $\hat{\rho}_x$ and $\hat{\rho}_y$ of $\hat{\boldsymbol{\rho}}$ in the same way.

d) Introduce dimensionless operators

$$\hat{a} = \frac{1}{\sqrt{2}l_B}(\hat{X} + i\hat{Y}) \,, \quad \hat{b} = \frac{1}{\sqrt{2}l_B}(\hat{\rho}_x - i\hat{\rho}_y) \tag{4}$$

where l_B is the so-called magnetic length, $l_B = \sqrt{\hbar/|eB|}$. Show that the set of operators $\{\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger\}$ satisfy the same commutation algebra as that of two independent harmonic oscillators. The corresponding set of harmonic oscillator states we denote by $|m, n\rangle$, where $\hat{a}^\dagger$ acts as a raising operator on the m quantum number and $\hat{b}^\dagger$ as a raising operator on the n quantum number.

e) Find expressions for the Hamiltonian $\hat{H}$ and angular momentum $\hat{L}$ in terms of the $\hat{a}$ and $\hat{b}$ operators. Show that $\hat{H}$ has an harmonic oscillator spectrum and find also the eigenvalues of $\hat{L}$ expressed in terms of m and n.

In the following we assume the particle to be restricted to the *degenerate* ground state of $\hat{H}$. For a charged particle in a magnetic field this is referred to as the lowest *Landau level*. Show that this restriction corresponds to the condition $n = 0$, while m is a free variable, so that the states $|m\rangle \equiv |m, 0\rangle, m = 0, 1, 2, ...$ form a complete set.

f) A coherent state in the Lowest Landau level can be defined by the equation

$$\hat{a}|z\rangle = z|z\rangle \,, \quad \hat{b}|z\rangle = 0 \tag{5}$$

Calculate the expectation values of the components of the position operator, $\hat{x}$ and $\hat{y}$ in the coherent state and show that it is peaked around the point $x = \sqrt{2}l_B \,\mathrm{Re}\, z, y = \sqrt{2}l_B \,\mathrm{Im}\, z$ in the x, y-plane. Use the coherent state representation for the $|m\rangle$ states, to demonstrate that the number of independent states in the lowest Landau level increases linearly with the available area in the x, y-plane. Find the density of states in the x, y-plane.

g) We assume that a weak, constant electric field E is introduced in the x-direction. Show that this effectively introduces the following Hamiltonian in the lowest Landau level,

$$H' = \frac{1}{2}\hbar\omega - \frac{l_b}{\sqrt{2}}eE(\hat{a} + \hat{a}^\dagger) \tag{6}$$

Also show that this Hamiltonian gives a time dependence to the coherent state $|z(t)\rangle$, corresponding to a drift with constant velocity in the y-direction. What is the drift velocity?