Problem set 12

12.1 Photon emission

A particle with mass m and charge e is trapped in a one-dimensional harmonic oscillator potential, with the motion restricted to the z-axis. The frequency of the oscillator is ω. At time $t = 0$ the particle is excited to energy level n and it then performs a transition to level $n-1$ by emitting one photon of energy $\hbar\omega$. We write the energy eigenstates of the composite system of charged particle and photons as $|n, n_{\mathbf{k}a}\rangle$. With initially no photon present the state is $|i\rangle = |n, 0\rangle$, while the final state with one photon present is $|f\rangle = |n-1, 1_{\mathbf{k}a}\rangle$. To first order in perturbation theory the angular probability distribution $p(\theta, \phi)$ of the emitted photon is

$$p(\theta, \phi) = \kappa \sum_a |\langle n-1, 1_{\mathbf{k}a}|\hat{H}_{emis}|n, 0\rangle|^2 \tag{1}$$

with (θ, ϕ) as the polar angle of the photon quantum number $\mathbf{k}$ and κ as a proportionality factor. $\hat{H}_{emis}$ is the emission part of the interaction Hamiltonian, which in the dipole approximation is

$$\hat{H}_{emis} = -\frac{e}{m} \sum_{\mathbf{k}a} \sqrt{\frac{\hbar}{2V\epsilon_0\,\omega}}\, \hat{\mathbf{p}} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}a}\, \hat{a}^\dagger_{\mathbf{k}a} \tag{2}$$

a) Show that for an arbitrary (real) vector $\mathbf{a}$ we have the identity

$$\sum_a (\mathbf{a} \cdot \boldsymbol{\epsilon}_{\mathbf{k}a})^2 = \mathbf{a}^2 - (\mathbf{a} \cdot \frac{\mathbf{k}}{k})^2 \tag{3}$$

b) Determine the particle matrix element $\langle n-1|\hat{\mathbf{p}}|n\rangle$.

c) Find the probability distribution $p(\theta, \phi)$.

The relation between the momentum operator and the ladder operators of the harmonic oscillator is found in Sect. 1.4.4 of the lecture notes.

12.2 Electric dipole transition in hydrogen (Exam 2008)

We consider the transition in hydrogen from the excited 2p level to the ground state 1s, where a single photon is emitted. The initial atomic state (A) we assume to have $m = 0$ for the z-component of the orbital angular momentum, so that the quantum numbers of this state are $(n, l, m) = (2, 1, 0)$, with n as the principle quantum number and l as the orbital angular momentum quantum number. Similarly the ground state (B) has quantum numbers $(n, l, m) = (1, 0, 0)$. When expressed in polar coordinates the wave functions of the two states (with intrinsic spin of the electron not included) are given by

$$\begin{aligned}
\psi_A(r, \phi, \theta) &= \frac{1}{\sqrt{32\pi a_0^3}} \cos\theta\, \frac{r}{a_0}\, e^{-\frac{r}{2a_0}} \\
\psi_B(r, \phi, \theta) &= \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}
\end{aligned} \tag{4}$$

where a_0 is the Bohr radius.

We remind you about the form of the interaction matrix element in the dipole approximation,

$$\langle B, 1_{\mathbf{k}a}|\hat{H}_{emis}|A, 0\rangle = ie\sqrt{\frac{\hbar\omega}{2V\epsilon_0}}\,\boldsymbol{\epsilon}_{\mathbf{k}a}\cdot\mathbf{r}_{BA} \tag{5}$$

where e is the electron charge, $\mathbf{k}$ is the wave vector of the photon, a is the polarization quantum number, ω is the photon frequency and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is a polarization vector. V is a normalization volume for the electromagnetic wave functions, ϵ_0 is the permittivity of vacuum and $\mathbf{r}_{BA}$ is the matrix element of the electron position operator between the initial and final atomic states.

a) Explain why the x- and y-components of $\mathbf{r}_{BA}$ vanish while the z-component has the form $z_{BA} = \nu a_0$, with ν as a numerical factor. Determine the value of ν. (A useful integration formula is $\int_0^\infty dx\, x^n\, e^{-x} = n!$.)

b) To first order in perturbation theory the interaction matrix element (5) determines the direction of the emitted photon, in the form of a probability distribution $p(\phi, \theta)$, where (ϕ, θ) are the polar angles of the wave vector $\mathbf{k}$. Determine $p(\phi, \theta)$ from the above expressions.

c) The life time of the 2p state is $\tau_{2p} = 1.6\cdot 10^{-9}s$ while the excited 2s state (with angular momentum $l = 0$) has a much longer life time, $\tau_{2s} = 0.12s$. Do you have a (qualitative) explanation for the large difference?

12.3 Spinflip radiation

We will study the transition between the two spin states of an electron in an external magnetic field directed along the z-axis, $\mathbf{B} = b\mathbf{e}_z$. The Hamiltonian can be expressed as $H = H_0 + H_1$, where H_0 descibes the energy of a magnetic dipole in the external field, while H_1 describes the interaction with the radiation field. Then we have

$$H_0 = \frac{\hbar}{2}\omega_B\sigma_z, \qquad \omega_B = -\frac{eB}{m}$$

where m is the electron mass and e the electron charge (which is negavive so that $\omega_B > 0$). The matrix element of the interaction pat H_1 for the case of emission if a single photon is in the dipole approximation given by

$$\langle B, 1_{\mathbf{k}a}|H_1|A, 0\rangle = i\frac{e\hbar}{2m}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}a})\cdot\boldsymbol{\sigma}_{BA}$$

where $|A\rangle$ is the excited spin state (spin up) and $|B\rangle$ is the ground state (spin down). $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is the polarization vector and $\omega = ck$ is the angular frequency of the emitted photon. V is the normalization volume for the electromagnetic radiation and $\boldsymbol{\sigma}_{BA} = \langle B|\boldsymbol{\sigma}|A\rangle$ is the matrix element of the vector $\boldsymbol{\sigma}$ of the Pauli matrices.

a) To first order in perturbation theory, the angular dependency of the squared matrix element $|\langle B, 1_{\mathbf{k}a}|H_1|A, 0\rangle|^2$ will determine the probability distribution for the direction of the emitted photon, $p(\theta, \phi)$, where (θ, ϕ) are the polar coordinates for the wavevector $\mathbf{k}$. Determine $p(\theta, \phi)$ using the above expression for the matrix element. It may be useful to know that when summing over the polarization states we have $\sum_a |\boldsymbol{\epsilon}_{\mathbf{k}a}\cdot\mathbf{b}|^2 = |\mathbf{b}|^2 - |\mathbf{b}\cdot\frac{\mathbf{k}}{k}|^2$ for an arbitrary vector b. The probability distribution should be normalized as $\int d\phi\int d\theta\sin\theta p(\theta, \phi) = 1$.

b) The squared matrix element also determines, for a given $\mathbf{k}$, the probability distribution for the polarization direction of the photon. Assume that a photon detector is set to detect photons emitted in the x-direction and with polarization vector $\boldsymbol{\epsilon}(\alpha) = \cos\alpha \mathbf{e}_y + \sin\alpha \mathbf{e}_z$ (here $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors in the x- and y- directions). Find the probability distribution $p(\alpha)$ to detect the emitted photon as a function of the angle α.

c) To a good approximation, the probability to find the spin in the excited state decays exponentielly with time

$$P_A(t) = e^{-t/\tau}$$

where the lifetime τ is, to first order in the interaction, determined by the time independent transition rate

$$w_{BA} = \frac{V}{(2\pi\hbar)^2} \int d^3k \sum_a |\langle B, 1_{\mathbf{k}a}|H_1|A, 0\rangle|^2 \delta(\omega - \omega_B)$$

Use this to find an expression for the lifetime τ.