Solutions to problem set 6

6.1 SWAP gate

a) We write $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$ and get

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) \\ &\overset{CNOT}{\to} a|0\rangle(c|0\rangle + d|1\rangle) + b|1\rangle(c|1\rangle + d|0\rangle) \\ &\overset{CNOT}{\to} ac|00\rangle + ad|11\rangle + bc|01\rangle + bd|10\rangle \\ &\overset{CNOT}{\to} ac|00\rangle + ad|10\rangle + bc|01\rangle + bd|11\rangle \\ &= (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle) = |\phi\rangle \otimes |\psi\rangle. \end{aligned}$$

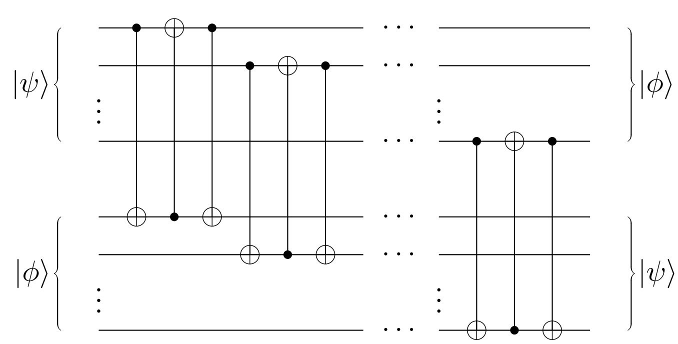
b) In the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the action of SWAP on the basis vectors is

$$|00\rangle \overset{SWAP}{\to} |00\rangle, \qquad |01\rangle \overset{SWAP}{\to} |10\rangle, \qquad |10\rangle \overset{SWAP}{\to} |01\rangle, \qquad |11\rangle \overset{SWAP}{\to} |11\rangle,$$

which gives the matrix

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) We can SWAP multi-qubit registers one qubit at a time



We need $3n$ CNOT gates.

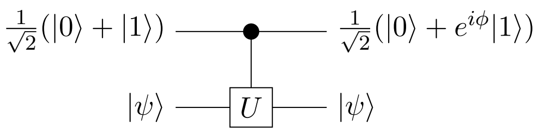
6.2 Quantum circuit for controlled R_k

a) We define $\phi = 2\pi/2^k$ and get

$$\begin{aligned}
|\psi_1\rangle \otimes |\psi_2\rangle &= (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle) \\
&\overset{R_{k+1}}{\to} (a_0|0\rangle + a_1 e^{i\phi/2}|1\rangle) \otimes (b_0|0\rangle + b_1 e^{i\phi/2}|1\rangle) \\
&\overset{CNOT}{\to} a_0|0\rangle \otimes (b_0|0\rangle + b_1 e^{i\phi/2}|1\rangle) + a_1 e^{i\phi/2}|1\rangle \otimes (b_0|1\rangle + b_1 e^{i\phi/2}|0\rangle) \\
&\overset{R^\dagger_{k+1}}{\to} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1 e^{i\phi/2}|1\rangle \otimes (b_0 e^{-i\phi/2}|1\rangle + b_1 e^{i\phi/2}|0\rangle) \\
&\overset{CNOT}{\to} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1|1\rangle \otimes (b_0|0\rangle + b_1 e^{i\phi}|1\rangle) \\
&= a_0|0\rangle \otimes |\psi_2\rangle + a_1|1\rangle \otimes R_k|\psi_2\rangle
\end{aligned}$$

This is the controlled R_k operation.

b) Let $U|\psi\rangle = e^{i\phi}|\psi\rangle$. The situation is described by this circuit



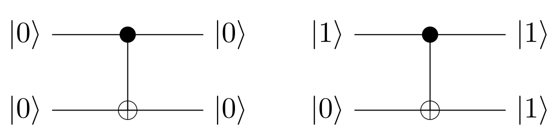
The evolution of the state is

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \otimes |\psi\rangle &\overset{\text{control}-U}{\to} \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle) \otimes \psi.
\end{aligned}$$

c) Since multiplying by a phase factor does not change a quantum state, U does not really change the state of the target if the initial state is an eigenstate. However, the relative phase between two states does make a physical difference. Therefore, when the control is in a superposition, there is a phase difference between the two states after the control-U operation. Since the state of the target is the same in both cases, it factors out, leaving a product state with the relative phase between the two states of the control qubit.

6.3 Quantum cloning of orthogonal states

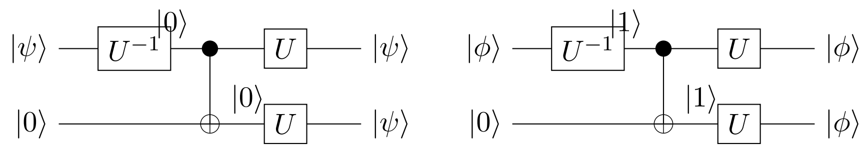
a) Assume first that $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |1\rangle$. Then we can check that a single CNOT gate gives the desired result (upper line is the original, lower line is the copy)



Since $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, there exist a unitary transformation U such that

$$|\psi\rangle = U|0\rangle$$
$$|\phi\rangle = U|1\rangle$$

The inverse of this transforms $|\psi\rangle$ and $|\phi\rangle$ to $|0\rangle$ and $|1\rangle$, and we can then use the CNOT as above and transform the result back, giving the final circuit



b) Here we can use the simple circuit with a single CNOT gate. The input is (qubits are written from top down)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

giving the final state

$$|\psi_1\rangle = C_{NOT}|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$