Solutions to problem set 9

9.1 Teleporting a unitary transformation

A teleports the state $|\psi\rangle$ to B, B performs the unitary operation U on the two qubits and teleports the resulting qubit back to A. Each teleportation requires one entangled pair and two bits of classical information.

The only worry is whether the entanglement between the two qbits that is created by the operation U will survive the teleportation. To convince oneself about this, one can use the Schmidt decomposition of the state. Let us call the state of the two qubits after U

$$|\chi\rangle_{14} = \sum_i d_i |\chi_i\rangle_1 \otimes |\phi_i\rangle_4$$

where we follow the convention in the lecture notes that the qubit to be teleported is number 1, the two qubits in the entangled pair is numbers 2 and 3, with 3 being the one to end in the teleported state. The qubit at B that is entangled with qubit 1 is number 4. At the end we want the entanglement to be transferred to number 3, which is at A. The teleportation protocol for one of the states $|\chi_i\rangle$ is (compare to Eqs. (3.11) and (3.16) in the lecture notes)

$$|\phi_i\rangle_{123} = |\chi_i\rangle_1 \otimes |\phi^-\rangle_{23} = \frac{1}{2}\sum_k |B_k\rangle_{12} \otimes V_k |\chi_i\rangle_3$$

where $|B_k\rangle_{12}$ are the Bell states defined in Eq (3.12) of the lecture notes. We then have

$$\begin{aligned}|\chi\rangle_{14} \otimes |\phi^-\rangle_{23} &= \sum_i d_i |\phi_i\rangle_{123} \otimes |\phi_i\rangle_4 = \frac{1}{2}\sum_i d_i \sum_k |B_k\rangle_{12} \otimes V_k |\chi_i\rangle_3 \otimes |\phi_i\rangle_4 \\ &= \frac{1}{2}\sum_k |B_k\rangle_{12} \otimes V_k \sum_i d_i |\chi_i\rangle_3 \otimes |\phi_i\rangle_4 = \frac{1}{2}\sum_k |B_k\rangle_{12} \otimes V_k |\chi\rangle_{34}\end{aligned}$$

We see that qubits 3 and 4 end in the desired entangled state if we apply the inverse of the V_k depending on the outcome of the measurement on qubits 1 and 2. Thus, the entanglement is transferred during teleportation.

9.2 Quantum gates for teleportation

a) We have to calculate the action of each gate on the state. The initial state is

$$|\psi_0\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes |0\rangle \otimes |0\rangle = c_0|000\rangle + c_1|100\rangle.$$

We write H^i for the Hadamard gate on qubit i, and C_{NOT}^{ij} for the CNOT gate with i as control bit and j as target bit. After each gate we then get

$$|\psi_1\rangle = H^b|\psi_0\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|010\rangle + c_1|100\rangle + c_1|110\rangle\Big]$$

$$|\psi_2\rangle = C_{NOT}^{bc}|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|011\rangle + c_1|100\rangle + c_1|111\rangle\Big]$$
$$|\psi_3\rangle = C_{NOT}^{ab}|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big[c_0|000\rangle + c_0|011\rangle + c_1|110\rangle + c_1|101\rangle\Big]$$

$$|\psi_4\rangle = H^a|\psi_3\rangle = \frac{1}{2}\Big[c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle\Big]$$

$$|\psi_5\rangle = C_{NOT}^{bc}|\psi_4\rangle = \frac{1}{2}\Big[c_0|000\rangle + c_0|100\rangle + c_0|010\rangle + c_0|110\rangle + c_1|011\rangle - c_1|111\rangle + c_1|001\rangle - c_1|101\rangle\Big]$$

$$|\psi_6\rangle = H^c|\psi_5\rangle = \frac{1}{2\sqrt{2}}\Big[(c_0+c_1)|000\rangle + (c_0-c_1)|001\rangle + (c_0-c_1)|100\rangle + (c_0+c_1)|101\rangle + (c_0+c_1)|010\rangle + (c_0-c_1)|011\rangle + (c_0-c_1)|110\rangle + (c_0+c_1)|111\rangle\Big]$$

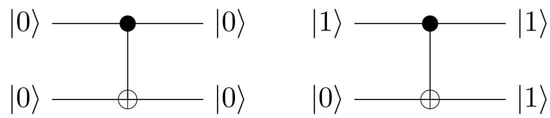
$$|\psi_7\rangle = C_{NOT}^{ac}|\psi_6\rangle = \frac{1}{2\sqrt{2}}\Big[(c_0+c_1)|000\rangle + (c_0-c_1)|001\rangle + (c_0-c_1)|101\rangle + (c_0+c_1)|100\rangle + (c_0+c_1)|010\rangle + (c_0-c_1)|011\rangle + (c_0-c_1)|111\rangle + (c_0+c_1)|110\rangle\Big]$$

$$\begin{aligned}|\psi_8\rangle = H^c|\psi_7\rangle &= \frac{1}{2}\Big[c_0|000\rangle + c_1|001\rangle + c_0|100\rangle + c_1|101\rangle + c_0|010\rangle + c_1|011\rangle + c_0|110\rangle + c_1|111\rangle\Big] \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (c_0|0\rangle + c_1|1\rangle)\end{aligned}$$

b) Measuring qubits a and b at the dashed line collapses the wavefunction at that point. But since a and b only acts as control bits forthe last four gates, their states do not change. Then the state will be the same as if we measure a and b on the final state $|\psi_8\rangle$ instead. The only difference is that now the CNOT gates will not be nonlocal two-qubit gates, but rather local one-qubit gates on qubit c conditioned on the measurement outcomes for a and b. This has to be transmitted from a and b to c as in the usual teleportation protocol. Then we still get $|c'\rangle = |a\rangle$ at the end. and only need local operations after the dashed line.

9.3 Quantum cloning of orthogonal states

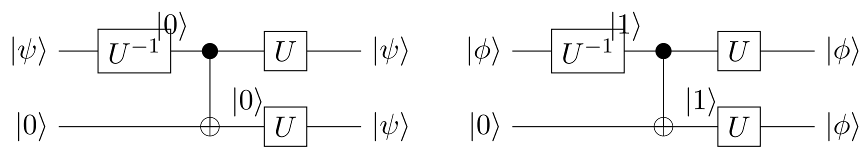
a) Assume first that $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |1\rangle$. Then we can check that a single CNOT gate gives the desired result (upper line is the original, lower line is the copy)



Since $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, there exist a unitary transformation U such that

$$|\psi\rangle = U|0\rangle$$
$$|\phi\rangle = U|1\rangle$$

The inverse of this transforms $|\psi\rangle$ and $|\phi\rangle$ to $|0\rangle$ and $|1\rangle$, and we can then use the CNOT as above and transform the result back, giving the final circuit



b) Here we can use the simple circuit with a single CNOT gate. The input is (qubits are written from top down)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

giving the final state

$$|\psi_1\rangle = C_{NOT}|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

9.4 See solution to Midterm exam 2020