

Solutions to problem set 8

8.1 SWAP gate

a) We write $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$ and get

$$\begin{aligned}
 |\psi\rangle \otimes |\phi\rangle &= (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) \\
 &\xrightarrow{CNOT} a|0\rangle(c|0\rangle + d|1\rangle) + b|1\rangle(c|1\rangle + d|0\rangle) \\
 &\xrightarrow{CNOT} ac|00\rangle + ad|11\rangle + bc|01\rangle + bd|10\rangle \\
 &\xrightarrow{CNOT} ac|00\rangle + ad|10\rangle + bc|01\rangle + bd|11\rangle \\
 &= (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle) = |\phi\rangle \otimes |\psi\rangle.
 \end{aligned}$$

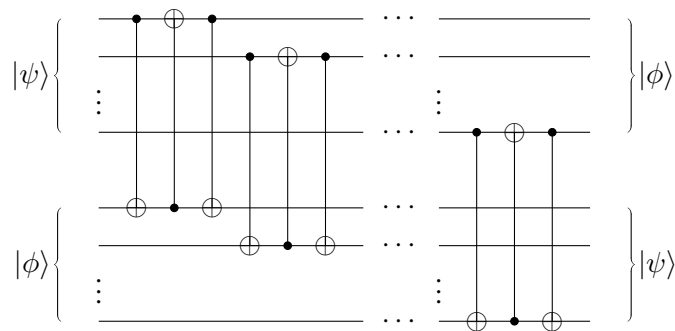
b) In the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the action of SWAP on the basis vectors is

$$|00\rangle \xrightarrow{SWAP} |00\rangle, \quad |01\rangle \xrightarrow{SWAP} |10\rangle, \quad |10\rangle \xrightarrow{SWAP} |01\rangle, \quad |11\rangle \xrightarrow{SWAP} |11\rangle,$$

which gives the matrix

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) We can SWAP multi-qubit registers one qubit at a time



We need $3n$ CNOT gates.

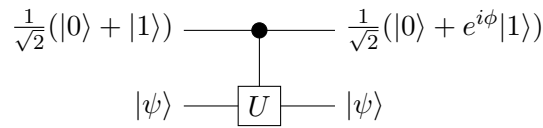
8.2 Quantum circuit for controlled R_k

a) We define $\phi = 2\pi/2^k$ and get

$$\begin{aligned}
 |\psi_1\rangle \otimes |\psi_2\rangle &= (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle) \\
 &\xrightarrow{R_{k+1}^\dagger} (a_0|0\rangle + a_1e^{i\phi/2}|1\rangle) \otimes (b_0|0\rangle + b_1e^{i\phi/2}|1\rangle) \\
 &\xrightarrow{CNOT} a_0|0\rangle \otimes (b_0|0\rangle + b_1e^{i\phi/2}|1\rangle) + a_1e^{i\phi/2}|1\rangle \otimes (b_0|1\rangle + b_1e^{i\phi/2}|0\rangle) \\
 &\xrightarrow{R_{k+1}^\dagger} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1e^{i\phi/2}|1\rangle \otimes (b_0e^{-i\phi/2}|1\rangle + b_1e^{i\phi/2}|0\rangle) \\
 &\xrightarrow{CNOT} a_0|0\rangle \otimes (b_0|0\rangle + b_1|1\rangle) + a_1|1\rangle \otimes (b_0|0\rangle + b_1e^{i\phi}|1\rangle) \\
 &= a_0|0\rangle \otimes |\psi_2\rangle + a_1|1\rangle \otimes R_k|\psi_2\rangle
 \end{aligned}$$

This is the controlled R_k operation.

b) Let $U|\psi\rangle = e^{i\phi}|\psi\rangle$. The situation is described by this circuit



The evolution of the state is

$$\begin{aligned}
 \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle &\xrightarrow{\text{control}-U} \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle) \\
 &= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle) \otimes \psi.
 \end{aligned}$$

c) Since multiplying by a phase factor does not change a quantum state, U does not really change the state of the target if the initial state is an eigenstate. However, the relative phase between two states does make a physical difference. Therefore, when the control is in a superposition, there is a phase difference between the two states after the control- U operation. Since the state of the target is the same in both cases, it factors out, leaving a product state with the relative phase between the two states of the control qubit.

8.3 Approximate quantum cloning

a) The state after the action of the operator U is

$$U|\psi\rangle_A|00\rangle_{BC} = \sqrt{\frac{2}{3}}(\alpha|000\rangle + \beta|111\rangle) + \sqrt{\frac{1}{6}}(\alpha|011\rangle + \alpha|101\rangle + \beta|010\rangle + \beta|100\rangle).$$

This gives the density matrix

$$\rho = \left[\sqrt{\frac{2}{3}} (\alpha|000\rangle + \beta|111\rangle) + \sqrt{\frac{1}{6}} (\alpha|011\rangle + \alpha|101\rangle + \beta|010\rangle + \beta|100\rangle) \right] \\ \left[\sqrt{\frac{2}{3}} (\alpha\langle 000| + \beta\langle 111|) + \sqrt{\frac{1}{6}} (\alpha\langle 011| + \alpha\langle 101| + \beta\langle 010| + \beta\langle 100|) \right].$$

The reduced density matrix of system A is then

$$\rho_A = \frac{2}{3} (|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + \alpha\beta^*|0\rangle\langle 1| + \beta\alpha^*|1\rangle\langle 0|) + \frac{1}{6}\mathbb{I} = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{6}\mathbb{I}.$$

The density matrix ρ is symmetric in the A and B systems, so ρ_B has the same form.

b) The initial state $|\psi\rangle$ has Bloch vector $\mathbf{m}^{(0)}$ given by

$$\rho_0 = |\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{I} + m_i^{(0)}\sigma_i).$$

We can then find the final state Bloch vector $\mathbf{m}$ from

$$\rho_A = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{6}\mathbb{I} = \frac{1}{2}(\mathbb{I} + m_i\sigma_i)$$

which gives that $\mathbf{m} = \frac{2}{3}\mathbf{m}^{(0)}$. This applies to both systems A and B since their reduced density matrices are the same. We see that the Bloch vector of the final state is parallel to the Bloch vector of the initial state, but with reduced length.

c)

$$F = \langle\psi|\rho_B|\psi\rangle = \frac{5}{6}.$$

d) We assume that the initial preparation step $|\psi\rangle_A|00\rangle_{BC} \rightarrow |\psi\rangle_A|\psi_0\rangle_{BC}$ already is implemented and follow the state through the rest of the circuit, numbering the CNOT-gates from left

$$\begin{aligned} |\psi\rangle_A|\psi_0\rangle_{BC} &= \sqrt{\frac{2}{3}}\alpha|000\rangle + \sqrt{\frac{1}{6}}\alpha|001\rangle + \sqrt{\frac{1}{6}}\alpha|011\rangle + \sqrt{\frac{2}{3}}\beta|100\rangle + \sqrt{\frac{1}{6}}\beta|101\rangle + \sqrt{\frac{1}{6}}\beta|111\rangle \\ &\xrightarrow{CNOT_1} \sqrt{\frac{2}{3}}\alpha|000\rangle + \sqrt{\frac{1}{6}}\alpha|001\rangle + \sqrt{\frac{1}{6}}\alpha|011\rangle + \sqrt{\frac{2}{3}}\beta|110\rangle + \sqrt{\frac{1}{6}}\beta|111\rangle + \sqrt{\frac{1}{6}}\beta|101\rangle \\ &\xrightarrow{CNOT_2} \sqrt{\frac{2}{3}}\alpha|000\rangle + \sqrt{\frac{1}{6}}\alpha|001\rangle + \sqrt{\frac{1}{6}}\alpha|011\rangle + \sqrt{\frac{2}{3}}\beta|111\rangle + \sqrt{\frac{1}{6}}\beta|110\rangle + \sqrt{\frac{1}{6}}\beta|100\rangle \\ &\xrightarrow{CNOT_3} \sqrt{\frac{2}{3}}\alpha|000\rangle + \sqrt{\frac{1}{6}}\alpha|001\rangle + \sqrt{\frac{1}{6}}\alpha|111\rangle + \sqrt{\frac{2}{3}}\beta|011\rangle + \sqrt{\frac{1}{6}}\beta|010\rangle + \sqrt{\frac{1}{6}}\beta|100\rangle \\ &\xrightarrow{CNOT_4} \sqrt{\frac{2}{3}}\alpha|000\rangle + \sqrt{\frac{1}{6}}\alpha|101\rangle + \sqrt{\frac{1}{6}}\alpha|011\rangle + \sqrt{\frac{2}{3}}\beta|111\rangle + \sqrt{\frac{1}{6}}\beta|010\rangle + \sqrt{\frac{1}{6}}\beta|100\rangle \\ &= \sqrt{\frac{2}{3}}(\alpha|000\rangle + \beta|111\rangle) + \sqrt{\frac{1}{6}}(\alpha|011\rangle + \alpha|101\rangle + \beta|010\rangle + \beta|100\rangle). \end{aligned}$$