Average number of photons

The average number of photons at a given frequency ω (in the lecture notes is Eq. 5.48) can be derived from the probability $P(E_n)$ to have a photon occupying a state with energy $E_n = n\hbar\omega$, that is given by

$$P(E_n) = \frac{e^{-n\beta\hbar\omega}}{Z}, \tag{1}$$

where $Z = \sum_{n=0}^{\infty} e^{-nx} = (1-e^{-x})^{-1}$ with $x = \beta\hbar\omega$. Thus, the average number of photons follows as

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{\infty} nP(E_n) \\
&= \sum_{n=0}^{\infty} n\left(1-e^{-x}\right)e^{-nx} \\
&= \left(1-e^{-x}\right)\left(-\frac{d}{dx}\sum_{n=0}^{\infty} e^{-nx}\right) \\
&= \left(1-e^{-x}\right)\frac{e^{-x}}{\left(1-e^{-x}\right)^2} \\
&= \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega}-1}.
\end{aligned} \tag{2}$$