Problem set 10 for the course FYS4130

April 5, 2013

Mean field

1. Consider the Ising model with spins $\sigma = \pm 1$ on a $d-$dimensional lattice with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i, \tag{1}$$

   where B is the external magnetic field and $\langle ij \rangle$ represents summation over nearest neighbors ($\sum_{\langle i,j \rangle} = \frac{1}{2} \sum_i \sum_{j=n.n.(i)}$).

   - Determined the Gibbs and Hemholtz free energies, $G(T,B)$ and $F(T,M)$, in the mean field approximation using that $\sigma_i = m + \delta\sigma_i$, with $\delta\sigma_i$ beging a small fluctuation around the average value m.
   - Plot $G(T,B)$ and $F(T,M)$ for different temperatures below and above the critical temperature T_c.

Random walk

1. Consider a 1d random walk where the displacement is a continuous, random variable determined by the *probability density* $w(s_i)$ such that $w(s_i)\,ds_i$ is the probability to find the step length between s_i and $s_i + ds_i$. In the following let us for simplicity assume that the probabilities for each step are *the same*.

   - Express the average $\bar{x}$ and dispersion $\overline{(\Delta x)^2}$ through the single-step probability density $w(s)$.
   - Apply these general expressions for the case of a one-dimensional random walk with the step length l. Write the Gaussian probability density for this case.

2. Express the general probability density $\mathcal{P}(x)$ through $w(s)$. Show that if one defines the *Fourier component* $\tilde{\mathcal{P}}(k)$ and $\tilde{w}(k)$ of the probability densities $\mathcal{P}(x)$ and $w(s)$, respectively, as

$$\tilde{A}(k) = \int_{-\infty}^{\infty} A(x)\, e^{-ikx} dx\,, \quad A(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(k)\, e^{-ikx} dk\,,$$

   then

$$\tilde{P}(k) = \tilde{w}^N(k)\,. \tag{2}$$

   Hint: take into account the constrain $x = \sum_{i=1}^{N} s_i$ and express it using Dirac delta-function.

3. Apply Eq. (2) to the case of equal step length l discussed above.