Problem set 8 for the course FYS4130

March 7, 2013

Phonons and photons are bosons

taken from Statistical Mechanics: Entropy, Order parameter, and Complexity by James P. Sethna, Oxford University Press 2006. Phonons and photons are the elementary, harmonic excitations of the elastic and electromagnetic fields. Phonons are decoupled harmonic oscillators, with a distribution of frequencies ω. Similarly, the Hamiltonian of the electromagnetic field can be decomposed into harmonic normal modes called photons. This exercise will explain why we think of phonons and photons as particles, instead of excitations of harmonic modes.

(a) Show that the canonical partition function for a quantum harmonic oscillator of frequency ω is the same as the grand canonical partition function for bosons multiply filling a single state with energy $\hbar\omega$, with $\mu = 0$ (apart from a shift in the arbitrary zero of the total energy of the system).

The Boltzmann filling of a harmonic oscillator is therefore the same as the Bose-Einstein filling of bosons into a single quantum state, except for an extra shift in the energy of $\hbar\omega/2$. This extra shift is called the zero-point energy. The excitations within the harmonic oscillator are thus often considered as particles with Bose statistics: the nth excitation is n bosons occupying the oscillator's quantum state. This particle analogy becomes even more compelling for systems like phonons and photons where there are many harmonic oscillator states labeled by a wavevector k. Real, massive Bose particles like He^4 in free space have single-particle quantum eigenstates with a dispersion relation $\varepsilon_k = \hbar^2 k^2/2m$. Phonons and photons have one harmonic oscillator for every k, with an excitation energy $\varepsilon_k = \hbar\omega_k$. If we treat them, as in part (a), as bosons filling these as single-particle states we find that they are completely analogous to ordinary massive particles. (Photons even have the dispersion relation of a massless boson. If we take the mass to zero of a relativistic particle, $\varepsilon = \sqrt{m^2c^4 - \mathbf{p}^2c^2} \to |\mathbf{p}|c = \hbar c|\mathbf{k}|$.)

(b) Do phonons or photons Bose condense at low temperatures? Can you see why not? Can you think of a non-equilibrium Bose condensation of photons, where a macroscopic occupation of a single frequency and momentum state occurs?

Bose condensation in a band

taken from Statistical Mechanics: Entropy, Order parameter, and Complexity by James P. Sethna, Oxford University Press 2006. The density of single-particle eigenstates $g(E)$ of a system of non-interacting bosons forms a band; the eigen-energies are confined to a range $E_{min} < E < E_{max}$, so $g(E)$ is non-zero in this range and zero otherwise. The system is filled with a finite density of bosons. Which of the following is necessary for the system to undergo Bose condensation at low temperatures?

(a) $\frac{g(E)}{e^{\beta(E-E_{min})}+1}$ is finite as $E \to E_{min}^-$.

(b) $\frac{g(E)}{e^{\beta(E-E_{min})}-1}$ is finite as $E \to E_{min}^-$.

(c) $E_{min} \geq 0$.

(d) $\int_{E_{min}}^{E} dE' \frac{g(E')}{E'-E_{min}}$ is a convergent integral at the lower limit E_{min}.
(e) Bose condensation cannot occur in a system whose states are confined to an energy band.