UNIVERSITY OF OSLO

Department of physics

Compulsory assignment: FYS4130 Statistical physics

Date: Hand in by April 9

All symbols that are not explicitly defined are understood to have the definition given in the Lecture notes. The set contains two pages.

We will study the distribution of a variable $X = \sum_{i=1}^{N} x_i$ where the random increments x_i are distributed according to two different distributions, a flat one and a power law $P(x_i) \sim 1/x_i$.

Part 1 Most random number generators deliver numbers in the interval $0 \leq x_i \leq 1$ according to a flat, or constant distribution, which, by the replacement $x \to 2x - 1$ will satisfy the distribution

$$P_0(x) = \begin{cases} 1/2 & \text{when} \quad -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

1. Show that $P_0(x)$ is normalized and calculate $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$.
2. Write a code that generates a sequence of X values and calculate the variance $\langle \Delta X^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$. Find out how $\langle \Delta X^2 \rangle$ scales with N.
3. Make a histogram of the X-values and show that $P(X)$ satisfies the central limit theorem.

Part 2

The central limit theorem explains why so-called normal distributions, or Gaussians, are commonplace. Log-normal distributions are Gaussians in the variable $\ln x$ rather than x, and these distributions arise when x is a product- rather than a sum of random variables. A reasonable approximation to the log-normal distribution turns out to be the power law

$$P(x) \propto \frac{1}{x} \, , \tag{2}$$

and we shall examine its properties in what folllows. In order to study this distribution we first have to establish some properties of it and the way to implement it numerically.

1. Explain that this distribution is not normalizable and that cut-off values must be introduced for both small and large x values.

2. If y is picked from the flat distribution $P_0(y)$, then show that the dependent variable $x = x(y)$ is distributed as

$$P(x) = \frac{P_0(y)}{x'(y)}. \tag{3}$$

3. Take $x = Be^{Ay}$ and show that we can write

$$P(x) = \frac{1}{2Ax} \text{ when } B\,e^{-A} < x < B\,e^{A} \text{ and 0 otherwise.} \tag{4}$$

4. Show that this distribution is normalized and calculate $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle \Delta x^2 \rangle$.

5. Verify that the variance $\langle \Delta x^2 \rangle$ can be written as

$$\langle \Delta x^2 \rangle = (A\,\text{cotanh}(A) - 1)\,\frac{B^2}{A^2}\sinh^2(A) \tag{5}$$

so that we are free to specify $\langle \Delta x^2 \rangle$ and $\langle x \rangle$ through A and B.

6. Set $A = B = 1$, $N = 1000$ and calculate a sequence of X-values, the mean $\langle \Delta X \rangle$, and the variance $\langle \Delta X^2 \rangle$. Compare the theoretical $N\langle \Delta x^2 \rangle$ with $\langle \Delta X^2 \rangle$, and plot the expected Gaussian along with a histogram of the X-values.

7. Do the same with the parameter values $A = 4$, $B = N = 100$ and discuss the result.

8. Use the parameter values $A = 6$, $B = N = 100$ and calculate the running average of the X-histogram with an averaging window ΔX =10. Many plotting programs, such as xmgr or xmgrace has this routine built in. Now, check if the histogram is indeed symmetric around the mean like a Gaussian is. Does the central limit theorem apply in this case? Discuss the result.

9. Consider a random walk where each step follows $P(x)$ from equation (4), and where the steps are independent. Find mean-square displacement $\langle \Delta x^2 \rangle$ and the diffusion coefficient D. How does D scale with the parameter A?