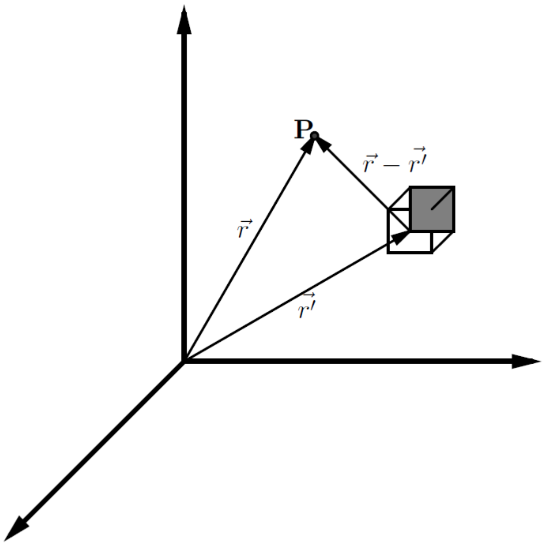
Lecture Notes in Physics 2018

Lecture 1. 150118



Let P be a point in the field (see figure 1.2) with position vector $\vec{r} = x^i\vec{e}_i$ and let the gravitating point source be at $\vec{r'} = x^{i'}\vec{e}_{i'}$. Newton's law of gravitation for a continuous distribution of mass is

$$\begin{aligned} \vec{F} &= -mG\int_r^\infty \rho(\vec{r'})\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}d^3r' \\ &= -\nabla V(\vec{r}) \end{aligned} \tag{1.4}$$

Note that the ∇-operator acts on the unprimed coordinates, only.

Let us consider equation (1.4) term by term

$$\begin{aligned} \nabla\frac{1}{|\vec{r}-\vec{r}'|} &= \vec{e}_i\frac{\partial}{\partial x_i}\frac{1}{\left[\left(x^j-x'^j\right)\left(x_j-x'_j\right)\right]^{1/2}} = \vec{e}_i\frac{\partial}{\partial x_i}\left[\left(x^j-x'^j\right)\left(x_j-x'_j\right)\right]^{-1/2} \\ &= \vec{e}_i\left(-\frac{1}{2}\right)\left[\left(x^j-x'^j\right)\left(x_j-x'_j\right)\right]^{-3/2}2\left(x^k-x'^k\right)\frac{\partial x_k}{\partial x_i} = -\vec{e}_i\frac{\left(x^k-x'^k\right)\delta^i_k}{\left[\left(x^j-x'^j\right)\left(x_j-x'_j\right)\right]^{3/2}} \\ &= -\frac{\left(x^i-x'^i\right)\vec{e}_i}{\left[\left(x^j-x'^j\right)\left(x_j-x'_j\right)\right]^{3/2}} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \end{aligned} \tag{1.5}$$

Now equations (1.4) and (1.5) together $\Rightarrow$

$$V(\vec{r}) = -mG \int \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|} d^3r' \tag{1.6}$$

Gravitational potential at point P:

$$\begin{aligned}
\phi(\vec{r}) \equiv \frac{V(\vec{r})}{m} &= -G \int \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|} d^3r' \\
\Rightarrow \quad \nabla\phi(\vec{r}) &= G \int \rho(\vec{r'}) \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} d^3r' \\
\Rightarrow \quad \nabla^2\phi(\vec{r}) &= G \int \rho(\vec{r'}) \nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} d^3r'
\end{aligned} \tag{1.7}$$

The above equation simplifies considerably if we calculate the divergence in the integrand.

$$\begin{aligned}
\nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} &= \frac{\nabla \cdot \vec{r}}{|\vec{r}-\vec{r'}|^3} + (\vec{r}-\vec{r'}) \cdot \nabla \frac{1}{|\vec{r}-\vec{r'}|^3} \\
&= \frac{3}{|\vec{r}-\vec{r'}|^3} - (\vec{r}-\vec{r'}) \cdot \frac{3(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^5} \\
&= \frac{3}{|\vec{r}-\vec{r'}|^3} - \frac{3}{|\vec{r}-\vec{r'}|^3} \\
&= 0 \quad \forall \quad \vec{r} \neq \vec{r'}
\end{aligned} \tag{1.8}$$

We conclude that the Newtonian gravitational potential at a point in a gravitational field outside a mass distribution satisfies Laplace's equation

$$\boxed{\nabla^2\phi = 0} \tag{1.9}$$