Lecture 14. 270218

Example 4.1.1 (Outer product of 1-forms in 3-space)

$$\underline{\alpha} = \alpha_i \underline{d}x^i \qquad x^i = (x, y, z)$$
$$\underline{d\alpha} = \alpha_{i,j} \underline{d}x^j \wedge \underline{d}x^i \tag{4.10}$$

Also, assume that $\underline{d\alpha} = 0$. The corresponding component equation is

$$\alpha_{[i,j]} = 0 \qquad \Rightarrow \qquad \alpha_{i,j} - \alpha_{j,i} = 0$$
$$\Rightarrow \frac{\partial \alpha_x}{\partial y} - \frac{\partial \alpha_y}{\partial x} = 0, \quad \frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x} = 0, \quad \frac{\partial \alpha_y}{\partial z} - \frac{\partial \alpha_z}{\partial y} = 0 \tag{4.11}$$

which corresponds to

$$\nabla \times \vec{\alpha} = 0 \tag{4.12}$$

The outer product of an outer product!

$$d^2\underline{\alpha} \equiv \underline{d}(\underline{d\alpha})$$
$$d^2\underline{\alpha} = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p, \nu_1 \nu_2} \underline{d}x^{\nu_2} \wedge \underline{d}x^{\nu_1} \wedge \cdots \wedge \underline{d}x^{\mu_p} \tag{4.13}$$

$$\boxed{{}_{,\nu_1 \nu_2} \equiv \frac{\partial^2}{\partial x^{\nu_1} \partial x^{\nu_2}}} \tag{4.14}$$

Since

$$,_{\nu_1\nu_2} \equiv \frac{\partial^2}{\partial x^{\nu_1}\partial x^{\nu_2}} =,_{\nu_2\nu_1} \equiv \frac{\partial^2}{\partial x^{\nu_2}\partial x^{\nu_1}} \tag{4.15}$$

summation over ν_1 and ν_2 which are symmetric in $\alpha_{\mu_1\cdots\mu_p,\nu_1\nu_2}$ and antisymmetric in the basis we get **Poincaré's lemma** (valid only for **scalar fields**)

$$\boxed{\underline{d}^2\underline{\alpha} = 0} \tag{4.16}$$

This corresponds to the vector equation

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{4.17}$$

Let $\underline{\alpha}$ be a p-form and $\underline{\beta}$ be a q-form. Then

$$\underline{d}(\underline{\alpha} \wedge \underline{\beta}) = \underline{d\alpha} \wedge \underline{\beta} + (-1)^p \underline{\alpha} \wedge \underline{d\beta} \tag{4.18}$$

4.1.2 Covariant derivative

The general theory of relativity contains a **covariance principle** which states that all equations expressing laws of nature must have the same form irrespective of the coordinate system in which they are derived. This is achieved by writing all equations in terms of tensors. Let us see if the partial derivative of vector components transform as tensor components. Given a vector $\vec{A} = A^\mu \vec{e}_\mu = A^{\mu'}\vec{e}_{\mu'}$ with the transformation of basis given by

$$\frac{\partial}{\partial x^{\nu\prime}} = \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu} \tag{4.19}$$

So that

$$\begin{aligned} A^{\mu'}{}_{,\nu'} &\equiv \frac{\partial}{\partial x^{\nu\prime}}\left(A^{\mu'}\right) \\ &= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu}\left(A^{\mu'}\right) \\ &= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x^{\mu\prime}}{\partial x^\mu}A^\mu\right) \\ &= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial x^{\mu\prime}}{\partial x^\mu}A^\mu{}_{,\nu} + \frac{\partial x^\nu}{\partial x^{\nu\prime}}A^\mu\frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\mu} \end{aligned} \tag{4.20}$$

The first term corresponds to a tensorial transformation. The existence of the last term shows that $A^{\mu}{}_{,\nu}$ does not, in general, transform as the components of a tensor. Note that $A^{\mu}{}_{,\nu}$ will transform as a tensor under linear transformations such as the Lorentz transformations.

The partial derivative must be generalized such as to ensure that when it is applied to tensor components it produces tensor components.

$$\frac{d\mathbf{A}}{d\lambda} = \frac{d}{d\lambda}\left(A^{\mu}\mathbf{e}_{\mu}\right) = \frac{dA^{\mu}}{d\lambda}\mathbf{e}_{\mu} + A^{\mu}\frac{d\mathbf{e}_{\mu}}{d\lambda} = A^{\mu}{}_{,\nu}u^{\nu}\mathbf{e}_{\mu} + A^{\mu}u^{\nu}\mathbf{e}_{\mu,\nu}, \qquad (6.89)$$

where $u^{\mu} \equiv \frac{dx^{\mu}}{d\lambda}$. Hence the partial derivative $A^{\mu}{}_{,\nu}$ does only represent the change of the vector component A^{μ}, and not the whole vector.

$$\frac{d\mathbf{e}_{\mu}}{d\lambda} = \Gamma^{\alpha}{}_{\mu\nu}\frac{dx^{\nu}}{d\lambda}\mathbf{e}_{\alpha} = \Gamma^{\alpha}{}_{\mu\nu}u^{\nu}\mathbf{e}_{\alpha},$$

where the functions $\Gamma^{\alpha}{}_{\mu\nu}$ are *connection coefficients*. Hence, eq.(6.89) takes the form

$$\frac{d\mathbf{A}}{d\lambda} = \left(A^{\mu}{}_{,\nu} + \Gamma^{\mu}{}_{\nu\alpha}A^{\alpha}\right)u^{\nu}\mathbf{e}_{\mu}. \qquad (6.93)$$

The covariant derivative, $A^{\mu}{}_{;\nu}$, of the vector components A^{μ} are defined by

$$\frac{d\mathbf{A}}{d\lambda} \equiv A^{\mu}{}_{;\nu}u^{\nu}\mathbf{e}_{\mu}. \qquad (6.94)$$

Comparing with the previous equation we obtain

$$\boxed{A^{\mu}{}_{;\nu} \equiv A^{\mu}{}_{,\nu} + A^{\alpha}\Gamma^{\mu}{}_{\alpha\nu}.} \qquad (6.95)$$

This derivative represents the change of the whole vector $\mathbf{A}$, not only the components A^{μ}.

4.2 The Christoffel Symbols

The covariant derivative was introduced by Christoffel to be able to differentiate tensor fields. It is defined in coordinate basis by generalizing the partially derivative $A^{\mu}{}_{,\nu}$ to a derivative written as $A^{\mu}{}_{;\nu}$ and which transforms tensorially,

$$A^{\mu'}{}_{;\nu'} \equiv \frac{\partial x^{\mu'}}{\partial x^{\mu}} \cdot \frac{\partial x^{\nu}}{\partial x^{\nu'}} A^{\mu}{}_{;\nu}. \tag{4.21}$$

The covariant derivative of the contravariant vector components are written as:

$$A^{\mu}{}_{;\nu} \equiv A^{\mu}{}_{,\nu} + A^{\alpha}\Gamma^{\mu}{}_{\alpha\nu} \tag{4.22}$$

This equation defines the Christoffel symbols $\Gamma^{\mu}{}_{\alpha\nu}$, which are also called the "connection coefficients in coordinate basis". From the transformation formulae for the two first terms follows that the Christoffel symbols transform as:

$$\Gamma^{\alpha'}{}_{\mu'\nu'} = \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \Gamma^{\alpha}{}_{\mu\nu} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'} \partial x^{\nu'}} \tag{4.23}$$

The Christoffel symbols do not transform as tensor components. It is possible to cancel all Christoffel symbols by transforming into a locally Cartesian coordinate system which is co-moving in a locally non-rotating reference frame in free fall. Such coordinates are known as **Gaussian coordinates**.

In general relativity theory an inertial frame is defined as a non-rotating frame in free fall. The Christoffel symbols are 0 (zero) in a locally Cartesian coordinate system which is co-moving in a local inertial frame. Local Gaussian coordinates are indicated with a bar over the indices, giving

$$\Gamma^{\bar{\alpha}}{}_{\bar{\mu}\bar{\nu}} = 0 \tag{4.24}$$

A transformation from local Gaussian coordinates to any coordinates leads to:

$$\Gamma^{\alpha'}{}_{\mu'\nu'} = \frac{\partial x^{\alpha'}}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\alpha}}}{\partial x^{\mu'} \partial x^{\nu'}} \tag{4.25}$$

This equation shows that the Christoffel symbols are symmetric in the two lower indices, ie.

$$\Gamma^{\alpha'}{}_{\mu'\nu'} = \Gamma^{\alpha'}{}_{\nu'\mu'} \tag{4.26}$$