Lecture 19. 19. mars 2018

4.7 Connection coefficients $\Gamma^{\alpha}{}_{\mu\nu}$ and structure coefficients $c^{\alpha}{}_{\mu\nu}$ in a Riemannian (torsion free) space

The commutator of two vectors, $\vec{u}$ and $\vec{v}$, expressed by covariant directional derivatives is given by:

$$[\vec{u}, \vec{v}] = \nabla_{\vec{u}}\vec{v} - \nabla_{\vec{v}}\vec{u} \tag{4.91}$$

Let $\vec{u} = \vec{e_\mu}$ and $\vec{v} = \vec{e_\nu}$. We then have:

$$[\vec{e_\mu}, \vec{e_\nu}] = \nabla_\mu \vec{e_\nu} - \nabla_\nu \vec{e_\mu}. \tag{4.92}$$

Using the definitions of the connection and structure coefficients we get:

$$c^{\alpha}{}_{\mu\nu}\vec{e_\alpha} = (\Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu})\vec{e_\alpha} \tag{4.93}$$

Thus in a torsion free space

$$c^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu} \tag{4.94}$$

In **coordinate basis** we have

$$\vec{e_\mu} = \frac{\partial}{\partial x^\mu}\ , \quad \vec{e_\nu} = \frac{\partial}{\partial x^\nu} \tag{4.95}$$

4.8.2 Covariant differentiation of forms

Definition 4.8.2 (Covariant directional derivative of a one-form field)
Given a vector field $\vec{A}$ and a one-form field $\underline{\alpha}$, the covariant directional derivative of $\underline{\alpha}$ in the direction of the vector $\vec{u}$ is defined by:

$$(\nabla_{\vec{u}}\underline{\alpha})(\vec{A}\,) \equiv \nabla_{\vec{u}}[\underbrace{\underline{\alpha}(\vec{A}\,)}_{\alpha_\mu A^\mu}] - \underline{\alpha}(\nabla_{\vec{u}}\vec{A}) \tag{4.103}$$

Let $\underline{\alpha} = \underline{\omega}^\mu$ (basis form), $\underline{\omega}^\mu(\vec{e_\nu}) \equiv \delta^\mu{}_\nu$ and let $\vec{A} = \vec{e_\nu}$ and $\vec{u} = \vec{e_\lambda}$. We then have:

$$(\nabla_\lambda \underline{\omega}^\mu)(\vec{e_\nu}) = \nabla_\lambda[\underbrace{\underline{\omega}^\mu(\vec{e_\nu})}_{\delta^\mu{}_\nu}] - \underline{\omega}^\mu(\nabla_\lambda \vec{e_\nu}) \tag{4.104}$$

The covariant directional derivative ∇_λ of a constant scalar field is zero, $\nabla_\lambda \delta^\mu{}_\nu = 0$. We therefore get:

$$\begin{aligned}(\nabla_\lambda \underline{\omega}^\mu)(\vec{e_\nu}) &= -\underline{\omega}^\mu(\nabla_\lambda \vec{e_\nu}) \\ &= -\underline{\omega}^\mu(\Gamma^{\alpha}{}_{\nu\lambda}\vec{e_\alpha}) \\ &= -\Gamma^{\alpha}{}_{\nu\lambda}\underline{\omega}^\mu(\vec{e_\alpha}) \\ &= -\Gamma^{\alpha}{}_{\nu\lambda}\delta^\mu{}_\alpha \\ &= -\Gamma^{\mu}{}_{\nu\lambda}\end{aligned} \tag{4.105}$$

The contraction between a one-form and a basis vector gives the components of the one-form, $\underline{\alpha}(\vec{e_\nu}) = \alpha_\nu$. Equation (4.105) tells us that the ν-component of $\nabla_\lambda \underline{\omega}^\mu$ is equal to $-\Gamma^{\mu}{}_{\nu\lambda}$, and that we therefore have

$$\boxed{\nabla_\lambda \underline{\omega}^\mu = -\Gamma^{\mu}{}_{\nu\lambda}\underline{\omega}^\nu} \tag{4.106}$$

Equation (4.106) gives the directional derivatives of the basis forms. Using the product of differentiation gives

$$\begin{aligned}\nabla_\lambda \underline{\alpha} &= \nabla_\lambda(\alpha_\mu \underline{\omega}^\mu) \\ &= \nabla_\lambda(\alpha_\mu)\underline{\omega}^\mu + \alpha_\mu \nabla_\lambda \underline{\omega}^\mu \\ &= \vec{e_\lambda}(\alpha_\mu)\underline{\omega}^\mu - \alpha_\mu \Gamma^{\mu}{}_{\nu\lambda}\underline{\omega}^\nu\end{aligned} \tag{4.107}$$

Definition 4.8.3 (Covariant derivative of a one-form)
The covariant derivative of a one-form $\underline{\alpha} = \alpha_\mu \underline{\omega}^\mu$ is therefore given by Equation (4.108) below, when we let $\mu \to \nu$ in the first term on the right hand side in (4.107):

$$\boxed{\nabla_\lambda \underline{\alpha} = [\vec{e_\lambda}(\alpha_\nu) - \alpha_\mu \Gamma^\mu{}_{\nu\lambda}]\underline{\omega}^\nu} \tag{4.108}$$

The covariant derivative of the one-form components α_μ are denoted by $\alpha_{\nu;\lambda}$ and are defined by

$$\nabla_\lambda \underline{\alpha} \equiv \alpha_{\nu;\lambda} \underline{\omega}^\nu \tag{4.109}$$

It then follows that

$$\boxed{\alpha_{\nu;\lambda} = \vec{e_\lambda}(\alpha_\nu) - \alpha_\mu \Gamma^\mu{}_{\nu\lambda}} \tag{4.110}$$

It is worth to note that $\Gamma^\mu{}_{\nu\lambda}$ in Equation (4.110) are not Christoffel symbols. In coordinate basis we get:

$$\alpha_{\nu;\lambda} = \alpha_{\nu,\lambda} - \alpha_\nu \Gamma^\mu{}_{\lambda\nu} \tag{4.111}$$

where $\Gamma^\mu{}_{\lambda\nu} = \Gamma^\mu{}_{\nu\lambda}$ are Christoffel symbols.

4.8.3 Generalization for tensors of higher rank

Definition 4.8.4 (Covariant derivative of a tensor)
Let A and B be two tensors of arbitrary rank. The covariant directional derivative along a basis vector $\vec{e_\alpha}$ of a tensor $A \otimes B$ of arbitrary rank is defined by:

$$\nabla_\lambda (A \otimes B) \equiv (\nabla_\lambda A) \otimes B + A \otimes (\nabla_\lambda B) \tag{4.112}$$

We will use (4.112) to find the formula for the covariant derivative of the components of a tensor of rank 2:

$$\begin{aligned} \nabla_\alpha S &= \nabla_\alpha (S_{\mu\nu} \underline{\omega}^\mu \otimes \underline{\omega}^\nu) \\ &= (\nabla_\alpha S_{\mu\nu}) \underline{\omega}^\mu \otimes \underline{\omega}^\nu + S_{\mu\nu} (\nabla_\alpha \underline{\omega}^\mu) \otimes \underline{\omega}^\nu + S_{\mu\nu} \underline{\omega}^\mu \otimes (\nabla_\alpha \underline{\omega}^\nu) \\ &= (S_{\mu\nu,\alpha} - S_{\beta\nu} \Gamma^\beta{}_{\mu\alpha} - S_{\mu\beta} \Gamma^\beta{}_{\nu\alpha}) \underline{\omega}^\mu \otimes \underline{\omega}^\nu \end{aligned} \tag{4.113}$$

where $S_{\mu\nu,\alpha} = \vec{e}_\alpha (S_{\mu\nu})$. Defining the covariant derivative $S_{\mu\nu;\alpha}$ by

$$\nabla_\alpha S = S_{\mu\nu;\alpha} \underline{\omega}^\mu \otimes \underline{\omega}^\nu \tag{4.114}$$

we get

$$S_{\mu\nu;\alpha} = S_{\mu\nu,\alpha} - S_{\beta\nu} \Gamma^\beta{}_{\mu\alpha} - S_{\mu\beta} \Gamma^\beta{}_{\nu\alpha} \tag{4.115}$$

For the metric tensor we get

$$g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - g_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha} - g_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} \tag{4.116}$$

From

$$g_{\mu\nu} = \vec{e_\mu} \cdot \vec{e_\nu} \tag{4.117}$$

we get:

$$\begin{aligned} g_{\mu\nu,\alpha} &= (\nabla_\alpha \vec{e_\mu}) \cdot \vec{e_\nu} + \vec{e_\mu}(\nabla_\alpha \vec{e_\nu}) \\ &= \Gamma^{\beta}{}_{\mu\alpha}\vec{e_\beta} \cdot \vec{e_\nu} + \vec{e_\mu} \cdot \Gamma^{\beta}{}_{\nu\alpha}\vec{e_\beta} \\ &= g_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha} + g_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} \end{aligned} \tag{4.118}$$

This means that

$$g_{\mu\nu;\alpha} = 0 \tag{4.119}$$

So the metric tensor is a (covariant) constant tensor.

4.9 The Cartan connection

Definition 4.9.1 (Exterior derivative of a basis vector)

$$\underline{d}\vec{e}_\mu \equiv \Gamma^{\nu}{}_{\mu\alpha}\vec{e}_\nu \otimes \underline{\omega}^\alpha \tag{4.120}$$

Exterior derivative of a vector field:

$$\underline{d}\vec{A} = \underline{d}(\vec{e}_\mu A^\mu) = \vec{e}_\nu \otimes \underline{d}A^\nu + A^\mu \underline{d}\vec{e}_\mu \tag{4.121}$$

In arbitrary basis:

$$\underline{d}A^\nu = \vec{e}_\lambda(A^\nu)\underline{\omega}^\lambda \tag{4.122}$$

(In coordinate basis, $\vec{e}_\lambda(A^\nu) = \frac{\partial}{\partial x^\lambda}(A^\nu) = A^\nu_{,\lambda}$)
giving:

$$\begin{aligned} \underline{d}\vec{A} &= \vec{e}_\nu \otimes [\vec{e}_\lambda(A^\nu)\underline{\omega}^\lambda] + A^\mu \Gamma^{\nu}{}_{\mu\lambda}\vec{e}_\nu \otimes \underline{\omega}^\lambda \\ &= (\vec{e}_\lambda(A^\nu) + A^\mu \Gamma^{\nu}{}_{\mu\lambda})\vec{e}_\nu \otimes \underline{\omega}^\lambda \end{aligned} \tag{4.123}$$

$$\boxed{\underline{d}\vec{A} = A^\nu_{;\lambda}\vec{e}_\nu \otimes \underline{\omega}^\lambda} \tag{4.124}$$

Definition 4.9.2 (Connection forms $\underline{\Omega}^{\nu}{}_{\mu}$)
The connection forms $\underline{\Omega}^{\nu}{}_{\mu}$ are 1-forms, defined by:

$$\begin{aligned} &\underline{d}\vec{e}_\mu \equiv \vec{e}_\nu \otimes \underline{\Omega}^{\nu}{}_{\mu} \\ &\Gamma^{\nu}{}_{\mu\alpha}\vec{e}_\nu \otimes \underline{\omega}^\alpha = \vec{e}_\nu \otimes \Gamma^{\nu}{}_{\mu\alpha}\underline{\omega}^\alpha = \vec{e}_\nu \otimes \underline{\Omega}^{\nu}{}_{\mu} \end{aligned} \tag{4.125}$$

$$\boxed{\underline{\Omega}^{\nu}{}_{\mu} = \Gamma^{\nu}{}_{\mu\alpha}\underline{\omega}^\alpha} \tag{4.126}$$

The exterior derivatives of the components of the metric tensor:

$$\underline{d}g_{\mu\nu} = \underline{d}(\vec{e}_\mu \cdot \vec{e}_\nu) = \vec{e}_\mu \cdot \underline{d}\vec{e}_\nu + \vec{e}_\nu \cdot \underline{d}\vec{e}_\mu \tag{4.127}$$

where the meaning of the dot is defined as follows:

Definition 4.9.3 (Scalar product between vector and 1-form)
The scalar product between a vector $\vec{u}$ and a (vectorial) one form $\underline{A} = A^\mu{}_\nu \vec{e}_\mu \otimes \underline{\omega}^\nu$ is defined by:

$$\vec{u} \cdot \underline{A} \equiv u^\alpha A^\mu{}_\nu (\vec{e}_\alpha \cdot \vec{e}_\mu) \underline{\omega}^\nu \tag{4.128}$$

Using this definition, we get:

$$\begin{aligned} \underline{d}g_{\mu\nu} &= (\vec{e}_\mu \cdot \vec{e}_\lambda)\underline{\Omega}^\lambda{}_\nu + (\vec{e}_\nu \cdot \vec{e}_\gamma)\underline{\Omega}^\gamma{}_\mu \\ &= g_{\mu\lambda}\underline{\Omega}^\lambda{}_\nu + g_{\nu\gamma}\underline{\Omega}^\gamma{}_\mu \end{aligned} \tag{4.129}$$

Lowering an index gives

$$\underline{d}g_{\mu\nu} = \underline{\Omega}_{\mu\nu} + \underline{\Omega}_{\nu\mu} \tag{4.130}$$

In an orthonormal basis field there is Minkowski-metric:

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \tag{4.131}$$

which is constant. Then we have :

$$\underline{d}g_{\hat{\mu}\hat{\nu}} = 0 \Rightarrow \boxed{\underline{\Omega}_{\hat{\nu}\hat{\mu}} = -\underline{\Omega}_{\hat{\mu}\hat{\nu}}} \tag{4.132}$$

where we write $\underline{\Omega}_{\hat{\nu}\hat{\mu}} = \Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}}\underline{\omega}^{\hat{\alpha}}$. It follows that $\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} = -\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}$.
It also follows that

$$\begin{aligned} &\Gamma^{\hat{t}}{}_{\hat{i}\hat{j}} = -\Gamma_{\hat{t}\hat{i}\hat{j}} = \Gamma_{\hat{i}\hat{t}\hat{j}} = \Gamma^{\hat{i}}{}_{\hat{t}\hat{j}} \\ &\Gamma^{\hat{i}}{}_{\hat{j}\hat{k}} = -\Gamma^{\hat{j}}{}_{\hat{i}\hat{k}} \end{aligned} \tag{4.133}$$