Lecture 2. 160118

The Dirac delta function has the following properties:

1. $\delta(\vec{r} - \vec{r'}) = 0 \quad \forall \quad \vec{r} \neq \vec{r'}$

2. $\int \delta(\vec{r} - \vec{r'}) d^3 r' = 1$ when $\vec{r} = \vec{r'}$ is contained in the integration domain. The integral is identically zero otherwise.

3. $\int f(\vec{r'}) \delta(\vec{r} - \vec{r'}) d^3 r' = f(\vec{r})$

A calculation of the integral $\int \nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} d^3 r'$ which is valid also in the case where the field point is inside the mass distribution is obtained through the use of Gauss' integral theorem:

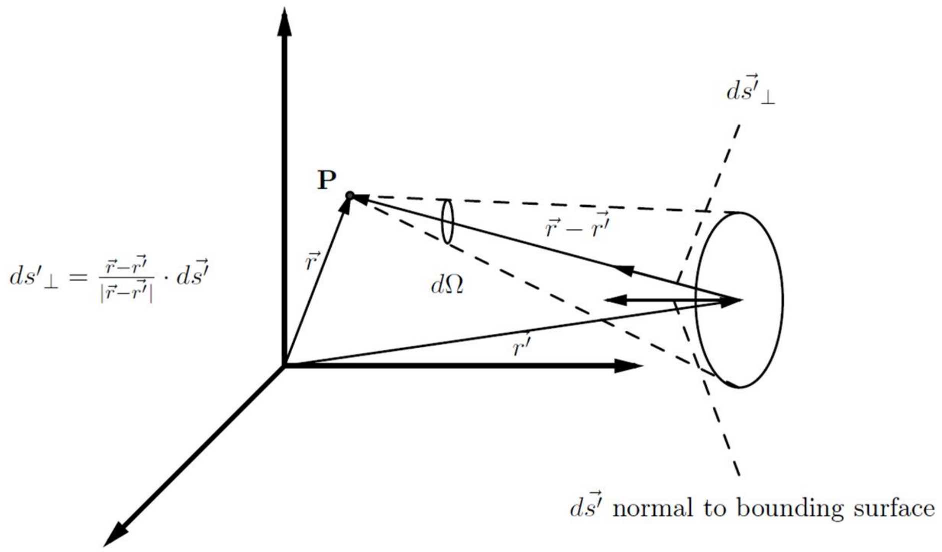
$$\int_{v} \nabla \cdot \vec{A} d^3 r' = \oint_{s} \vec{A} \cdot d\vec{s}, \tag{1.10}$$

where s is the boundary of v ($s = \partial v$ is an area).

Definition 1.2.1 (Solid angle)

$$d\Omega \equiv \frac{ds'_{\perp}}{|\vec{r} - \vec{r'}|^2} \tag{1.11}$$

where $ds'_{\perp}$ is the projection of the area ds' normal to the line of sight. $\vec{ds'}_{\perp}$ is the component vector of $\vec{ds'}$ along the line of sight which is equal to the normal vector of $ds'_{\perp}$ (see figure (1.3)).



Now, let's apply Gauss' integral theorem.

$$\int\limits_{\mathrm{v}} \nabla\cdot\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}d^3r' = \oint\limits_{\mathrm{s}} \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\cdot d\vec{s'} = \oint\limits_{\mathrm{s}} \frac{ds'_{\perp}}{|\vec{r}-\vec{r'}|^2} = \oint\limits_{\mathrm{s}} d\Omega \tag{1.12}$$

So that,

$$\int\limits_{\mathrm{v}} \nabla\cdot\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}d^3r' = \begin{cases} 4\pi & \text{if P is inside the mass distribution,} \\ 0 & \text{if P is outside the mass distribution.} \end{cases} \tag{1.13}$$

The above relation is written concisely in terms of the Dirac delta function:

$$\nabla\cdot\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} = 4\pi\delta(\vec{r}-\vec{r'}) \tag{1.14}$$

We now have

$$\begin{aligned} \nabla^2\phi(\vec{r}) &= G\int \rho(\vec{r'})\nabla\cdot\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}d^3r' \\ &= G\int \rho(\vec{r'})4\pi\delta(\vec{r}-\vec{r'})d^3r' \\ &= 4\pi G\rho(\vec{r}) \end{aligned} \tag{1.15}$$

Newton's theory of gravitation can now be expressed very succinctly indeed!

1. Mass generates gravitational potential according to

$$\nabla^2\phi = 4\pi G\rho \tag{1.16}$$

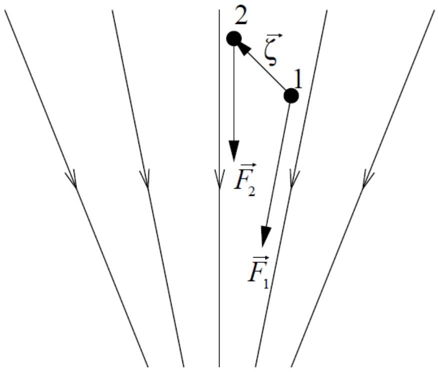
2. Gravitational potential generates motion according to

$$\vec{g} = -\nabla\phi \tag{1.17}$$

where $\vec{g}$ is the field strength of the gravitational field.

1.3 Tidal Forces

Tidal force is difference of gravitational force on two neighboring particles in a gravitational field. The tidal force is due to the inhomogeneity of a gravitational field.



In figure 1.4 two points have a separation vector ζ. The position vectors of 1 and 2 are $\vec{r}$ and $\vec{r}+\vec{\zeta}$, respectively, where $|\vec{\zeta}| \ll |\vec{r}|$. The gravitational forces on a mass m at 1 and at 2 are $\vec{F}(\vec{r})$ and $\vec{F}(\vec{r}+\vec{\zeta})$. By means of a Taylor expansion to lowest order in $|\vec{\zeta}|$ we get for the i-component of the tidal force

$$f_i = F_i(\vec{r}+\vec{\zeta}) - F_i(\vec{r}) = \zeta_j \left(\frac{\partial F_i}{\partial x^j}\right)_{\vec{r}}. \tag{1.18}$$

The corresponding vector equation is

$$\vec{f} = (\vec{\zeta}\cdot\nabla)_{\vec{r}}\vec{F}. \tag{1.19}$$

Using that

$$\vec{F} = -m\nabla\phi, \tag{1.20}$$

the tidal force may be expressed in terms of the gravitational potential according to

$$\vec{f} = -m(\vec{\zeta}\cdot\nabla)\nabla\phi. \tag{1.21}$$

Let us write this equation in component form,

$$\vec{f} = -m\left(\zeta^i \vec{e}_i \cdot \vec{e}_j \frac{\partial}{\partial x_j}\right)\frac{\partial\phi}{\partial x_k}\vec{e}_k. \tag{1.21.A}$$

Using an orthonormal basis so that

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}, \tag{1.21.B}$$

we obtain

$$\vec{f} = -m\zeta^i \frac{\partial^2 \phi}{\partial x_i \partial x_k} \vec{e}_k , \qquad (1.21.C)$$

or

$$f^k = -m\zeta^i \frac{\partial^2 \phi}{\partial x_i \partial x_k} . \qquad (1.21.D)$$

Combining with Newton's 2. law

$$f^k = m\frac{d^2 \zeta_k}{dt^2} \qquad (1.21.E)$$

gives the equation for the *tidal acceleration*, i.e. the relative acceleration between two nearby particles

$$\frac{d^2 \zeta^k}{dt^2} = -\zeta^i \frac{\partial^2 \phi}{\partial x_i \partial x_k} . \qquad (1.22)$$

Let us look at a few simple examples. In the first one $\vec{\zeta}$ has the same direction as $\vec{g}$. Consider a small Cartesian coordinate system at a distance R from a mass M (see figure 1.5). If we place a particle of mass m at a point $(0, 0, +z)$, it will, according to eq. (1.1) be acted upon by a force

$$F_z(+z) = -m\frac{GM}{(R+z)^2} \qquad (1.23)$$

while an identical particle at the origin will be acted upon by the force

$$F_z(0) = -m\frac{GM}{R^2}. \qquad (1.24)$$

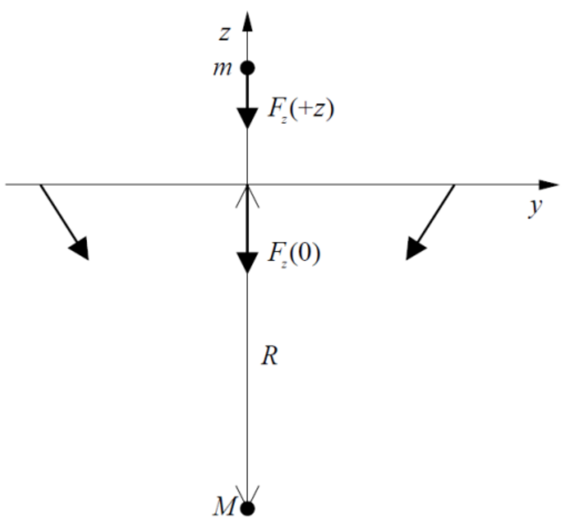


Figure 1.5: A small Cartesian coordinate system at a distance R from a mass M.

If this little coordinate system is falling freely towards M, an observer at the origin will say that the particle at $(0,0,z)$ is acted upon by a force

$$f_z = F_z(z) - F(0) = GmM\left[-\frac{1}{(R+z)^2} + \frac{1}{R^2}\right] = \frac{GmM}{R^2}\left[1 - \left(1+\frac{z}{R}\right)^{-2}\right]. \qquad (1.25A)$$

A Taylor expansion to second order in $x << 1$ is

$$(1+x)^p \approx 1 + px + (1/2)p(p-1)x^2 . \qquad (1.25B)$$

It is here sufficient to apply this to first order in $x = z/R$. This gives

$$f_z = 2GmM\frac{z}{R^3} . \qquad (1.25)$$

Hence the tidal force decreases with the third power of the distance from the body producing the gravitational field, i.e. faster that the ordinary gravitational force. We see that the tidal force implies a streching in the vertical direction.

In the same way particles at the points $(+x, 0, 0)$ and $(0, +y, 0)$ are attracted towards the origin by tidal forces

$$f_x = -mx\frac{GM}{R^3}, \qquad (1.26)$$

$$f_y = -my\frac{GM}{R^3}. \qquad (1.27)$$

Eqs. (1.25)–(1.27) have among others the following consequence: If an elastic, circular ring is falling freely in the Earth's gravitational field, as shown in figure 1.6, it will be stretched in the vertical direction and compressed in the horizontal direction.

In general, tidal forces cause changes of shape.