Lecture 24. 16. April 2018

6.2 Einstein's curvature tensor

The field equations are assumed to have the form:

space-time curvature $\propto$ momentum-energy tensor

Also, it is demanded that energy and momentum conservation should follow as a consequence of the field equation. This puts the following constraints on the curvature tensor: It must be a symmetric, divergence free tensor of rank 2.

Bianchi's 2nd identity:

$$R^{\mu}{}_{\nu\alpha\beta;\sigma} + R^{\mu}{}_{\nu\sigma\alpha;\beta} + R^{\mu}{}_{\nu\beta\sigma;\alpha} = 0 \tag{6.12}$$

contraction of μ and α $\Rightarrow$

$$\begin{aligned} R^{\mu}{}_{\nu\mu\beta;\sigma} - R^{\mu}{}_{\nu\mu\sigma;\beta} + R^{\mu}{}_{\nu\beta\sigma;\mu} &= 0 \\ R_{\ \nu\beta;\sigma} - R_{\ \nu\sigma;\beta} + R^{\mu}{}_{\nu\beta\sigma;\mu} &= 0 \end{aligned} \tag{6.13}$$

further contraction of ν and σ gives

$$\begin{aligned} R^{\sigma}{}_{\beta;\sigma} - R^{\sigma}{}_{\sigma;\beta} + R^{\sigma\mu}{}_{\sigma\beta;\mu} &= 0 \\ R^{\sigma}{}_{\beta;\sigma} - R_{\ ;\beta} + R^{\sigma}{}_{\beta;\sigma} &= 0 \\ \therefore \quad 2R^{\sigma}{}_{\beta;\sigma} &= R_{\ ;\beta} \end{aligned} \tag{6.14}$$

Thus, we have calculated the divergence of the Ricci tensor,

$$R^{\sigma}{}_{\beta;\sigma} = \frac{1}{2} R_{\ ;\beta} \tag{6.15}$$

Now we use this expression together with the fact that the metric tensor is covariant and divergence free to construct a new divergence free curvature tensor.

$$R^{\sigma}{}_{\beta;\sigma} - \frac{1}{2} R_{\ ;\beta} = 0 \tag{6.16}$$

Keeping in mind that $(g^{\sigma}{}_{\beta} R)_{;\sigma} = g^{\sigma}{}_{\beta} R_{;\sigma}$ we multiply (6.16) by $g^{\beta}{}_{\alpha}$ to get

$$\begin{aligned} g^{\beta}{}_{\alpha} R^{\sigma}{}_{\beta;\sigma} - g^{\beta}{}_{\alpha} \frac{1}{2} R_{\ ;\beta} &= 0 \\ \left(g^{\beta}{}_{\alpha} R^{\sigma}{}_{\beta} \right)_{;\sigma} - \frac{1}{2} \left(g^{\beta}{}_{\alpha} R \right)_{;\beta} &= 0 \end{aligned} \tag{6.17}$$

interchanging σ and β in the first term of the last equation:

$$\begin{aligned} \left(g^{\sigma}_{\ \alpha} R^{\beta}_{\ \sigma}\right)_{;\beta} - \frac{1}{2}\left(g^{\beta}_{\ \alpha} R\right)_{;\beta} &= 0 \\ \Rightarrow \quad \left(R^{\beta}_{\ \alpha} - \frac{1}{2}\delta^{\beta}_{\ \alpha} R\right)_{;\beta} &= 0 \end{aligned} \tag{6.18}$$

since $g^{\sigma}_{\ \alpha} R^{\beta}_{\ \sigma} = \delta^{\sigma}_{\ \alpha} R^{\beta}_{\ \sigma} = R^{\beta}_{\ \alpha}$. So that $R^{\beta}_{\ \alpha} - \frac{1}{2}\delta^{\beta}_{\ \alpha} R$ is the divergence free curvature tensor desired.

This tensor is called the Einstein tensor and its covariant components are denoted by $E_{\alpha\beta}$. That is

$$\boxed{E_{\alpha\beta} = R_{\ \alpha\beta} - \frac{1}{2} g_{\ \alpha\beta} R} \tag{6.19}$$

NOTE THAT: $E^{\mu\nu}_{\ ;\nu} = 0 \rightarrow 4$ equations, giving only 6 equations from $E_{\ \mu\nu}$, which secures a free choice of coordinate system.

6.3 Einstein's field equations

Einstein's field equations:

$$E_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.20}$$

or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \tag{6.21}$$

Contraction gives:

$$\begin{aligned} R - \frac{1}{2} 4R &= \kappa T \ , \qquad \text{where } T \equiv T^{\mu}_{\ \mu} \\ R &= -\kappa T \end{aligned} \tag{6.22}$$

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu}(-\kappa T) + \kappa T_{\mu\nu} \ , \tag{6.23}$$

Thus the field equations may be written in the form

$$R_{\mu\nu} = \kappa (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \tag{6.24}$$

In the Newtonian limit the metric may be written

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right) dt^2 + (1 + h_{ii})(dx^2 + dy^2 + dz^2) \tag{6.25}$$

where the Newtonian potential $|\phi| \ll c^2$. We also have $T_{00} \gg T_{kk}$ and $T \approx -T_{00}$. Then the 00-component of the field equations becomes

$$R_{00} \approx \frac{\kappa}{2} T_{00} \tag{6.26}$$

Furthermore we have

$$\begin{aligned} R_{00} = R^{\mu}{}_{0\mu 0} &= R^{i}{}_{0i0} \\ &= \Gamma^{i}{}_{00,i} - \Gamma^{i}{}_{0i,0} \\ &= \frac{\partial \Gamma^{k}{}_{00}}{\partial x^k} = \frac{1}{c^2} \nabla^2 \phi \end{aligned} \tag{6.27}$$

Since $T_{00} \approx \rho c^2$ eq.(6.26) can be written $\nabla^2 \phi = \frac{1}{2} \kappa c^4 \rho$. Comparing this equation with the Newtonian law of gravitation on local form: $\nabla^2 \phi = 4\pi G \rho$, we see that $\kappa = \frac{8\pi G}{c^4}$.

In classical vacuum we have : $T_{\mu\nu} = 0$, which gives

$$\boxed{E_{\mu\nu} = 0 \quad \text{or} \quad R_{\mu\nu} = 0\ .} \tag{6.28}$$

These are the "vacuum field equations". Note that $R_{\mu\nu} = 0$ does *not* imply $R_{\mu\nu\alpha\beta} = 0$.

Digression 6.3.1 (Lagrange (variation principle))
It was shown by Hilbert that the field equations may be deduced from a variation principle with action

$$\int R \sqrt{-g} d^4x\ , \tag{6.29}$$

where $R\sqrt{-g}$ is the Lagrange density. One may also include a so-called cosmological constant Λ:

$$\int (R + 2\Lambda) \sqrt{-g} d^4x \tag{6.30}$$

The field equations with cosmological constant are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.31}$$