Lecture 25. 17. april 2018

6.4 The "geodesic postulate" as a consequence of the field equations

The principle that free particles follow geodesic curves has been called the "geodesic postulate". We shall now show that the "geodesic postulate" follows as a consequence of the field equations.

Consider a system of free particles in curved space-time. This system can be regarded as a pressure-free gas. Such a gas is called *dust*. It is described by an energy-momentum tensor

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{6.32}$$

where ρ is the rest density of the dust as measured by an observer at rest in the dust and u^μ are the components of the four-velocity of the dust particles.

Einstein's field equations as applied to space-time filled with dust, take the form

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa \rho u^\mu u^\nu \tag{6.33}$$

Because the divergence of the left hand side is zero, the divergence of the right hand side must be zero, too

$$(\rho u^\mu u^\nu)_{;\nu} = 0 \tag{6.34}$$

or

$$(\rho u^\nu u^\mu)_{;\nu} = 0 \tag{6.35}$$

we now regard the quantity in the parenthesis as a product of ρu^ν and u^μ. By the rule for differentiating a product we get

$$(\rho u^\nu)_{;\nu} u^\mu + \rho u^\nu u^\mu{}_{;\nu} = 0 \tag{6.36}$$

Since the four-velocity of any object has a magnitude equal to the velocity of light we have

$$u_\mu u^\mu = -c^2 \tag{6.37}$$

Differentiation gives

$$(u_\mu u^\mu)_{;\nu} = 0 \tag{6.38}$$

Using, again, the rule for differentiating a product, we get

$$u_{\mu;\nu} u^\mu + u_\mu u^\mu{}_{;\nu} = 0 \tag{6.39}$$

From the rule for raising an index and the freedom of changing a summation index from α to μ, say, we get

$$u_{\ \mu;\nu} u^\mu = u^\mu u_{\mu;\nu} = g^{\mu\alpha} u_\alpha u_{\mu;\nu} = u_\alpha g^{\mu\alpha} u_{\mu;\nu} = u_\alpha u^\alpha{}_{;\nu} = u_\mu u^\mu{}_{;\nu} \tag{6.40}$$

Thus the two terms of eq.(6.39) are equal. It follows that each of them are equal to zero. So we have

$$u_\mu u^\mu{}_{;\nu} = 0 \tag{6.41}$$

Multiplying eq.(6.36) by u_μ, we get

$$(\rho u^\nu)_{;\nu} u_\mu u^\mu + \rho u^\nu u_\mu u^\mu{}_{;\nu} = 0 \tag{6.42}$$

Using eq.(6.37) in the first term, and eq.(6.41) in the last term, which then vanishes, we get

$$(\rho u^\nu)_{;\nu} = 0 \tag{6.43}$$

Thus the first term in eq.(6.36) vanishes and we get

$$\rho u^\nu u^\mu{}_{;\nu} = 0 \tag{6.44}$$

Since $\rho \neq 0$ we must have

$$u^\nu u^\mu{}_{;\nu} = 0 \tag{6.45}$$

This is just the geodesic equation. Conclusion: *It follows from Einstein's field equations that free particles move along paths corresponding to geodesic curves of space-time.*

7.1 Schwarzschild's exterior solution

This is a solution of the vacuum field equations $E_{\mu\nu} = 0$ for a static spherically symmetric spacetime. One can then *choose* the following form of the line element (employing units so that c=1),

$$\begin{aligned} ds^2 &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \\ d\Omega^2 &= d\theta^2 + \sin^2\theta d\phi^2 \end{aligned} \tag{7.1}$$

These coordinates are chosen so that the area of a sphere with radius r is $4\pi r^2$.

Physical distance in radial direction, corresponding to a coordinate distance dr, is $dl_r = \sqrt{g_{rr}}dr = e^{\beta(r)}dr$.

Here follows a stepwise algorithm to determine the components of the Einstein tensor by using the Cartan formalism:

1. Using orthonormal basis (ie. solving $E_{\hat{\mu}\hat{\nu}} = 0$) we find

$$\underline{\omega}^{\hat{t}} = e^{\alpha(r)}\underline{dt}\ , \quad \underline{\omega}^{\hat{r}} = e^{\beta(r)}\underline{dr}\ , \quad \underline{\omega}^{\hat{\theta}} = r\underline{d}\theta\ , \quad \underline{\omega}^{\hat{\phi}} = r\sin\theta\underline{d}\phi \tag{7.2}$$

2. Computing the connection forms by applying Cartan's 1. structure equations

$$\underline{d\omega}^{\hat{\mu}} = -\underline{\Omega}^{\hat{\mu}}_{\ \hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}} \tag{7.3}$$

$$\begin{aligned} \underline{d\omega}^{\hat{t}} &= e^{\alpha}\alpha'\underline{dr} \wedge \underline{dt} \\ &= e^{\alpha}\alpha' e^{-\beta}\underline{\omega}^{\hat{r}} \wedge e^{-\alpha}\underline{\omega}^{\hat{t}} \\ &= -e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}} \\ &= -\underline{\Omega}^{\hat{t}}_{\ \hat{r}} \wedge \underline{\omega}^{\hat{r}} \end{aligned} \tag{7.4}$$

$$\therefore \underline{\Omega}^{\hat{t}}_{\ \hat{r}} = e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} + f_1\underline{\omega}^{\hat{r}} \tag{7.5}$$

3. To determine the f-functions we apply the anti-symmetry

$$\underline{\Omega}_{\hat{\mu}\hat{\nu}} = -\underline{\Omega}_{\hat{\nu}\hat{\mu}} \tag{7.6}$$

This gives:

$$\begin{aligned}
\underline{\Omega}^{\hat{r}}{}_{\hat{\phi}} &= -\underline{\Omega}^{\hat{\phi}}{}_{\hat{r}} = -\frac{1}{r}e^{-\beta}\underline{\omega}^{\hat{\phi}} \\
\underline{\Omega}^{\hat{\theta}}{}_{\hat{\phi}} &= -\underline{\Omega}^{\hat{\phi}}{}_{\hat{\theta}} = -\frac{1}{r}\cot\theta\underline{\omega}^{\hat{\phi}} \\
\underline{\Omega}^{\hat{t}}{}_{\hat{r}} &= +\underline{\Omega}^{\hat{r}}{}_{\hat{t}} = e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} \\
\underline{\Omega}^{\hat{r}}{}_{\hat{\theta}} &= -\underline{\Omega}^{\hat{\theta}}{}_{\hat{r}} = -\frac{1}{r}e^{-\beta}\underline{\omega}^{\hat{\theta}}
\end{aligned} \tag{7.7}$$

4. We then proceed to determine the curvature forms by applying Cartan's 2nd structure equations

$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \underline{d\Omega}^{\hat{\mu}}{}_{\hat{\nu}} + \underline{\Omega}^{\hat{\mu}}{}_{\hat{\alpha}} \wedge \underline{\Omega}^{\hat{\alpha}}{}_{\hat{\nu}} \tag{7.8}$$

which gives:

$$\begin{aligned}
\underline{R}^{\hat{t}}{}_{\hat{r}} &= -e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta')\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}} \\
\underline{R}^{\hat{t}}{}_{\hat{\theta}} &= -\frac{1}{r}e^{-2\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}} \\
\underline{R}^{\hat{t}}{}_{\hat{\phi}} &= -\frac{1}{r}e^{-2\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\phi}} \\
\underline{R}^{\hat{r}}{}_{\hat{\theta}} &= \frac{1}{r}e^{-2\beta}\beta'\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\theta}} \\
\underline{R}^{\hat{r}}{}_{\hat{\phi}} &= \frac{1}{r}e^{-2\beta}\beta'\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\phi}} \\
\underline{R}^{\hat{\theta}}{}_{\hat{\phi}} &= \frac{1}{r^2}(1 - e^{-2\beta})\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}}
\end{aligned} \tag{7.9}$$

5. By applying the following relation

$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \frac{1}{2}R^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}}\underline{\omega}^{\hat{\alpha}} \wedge \underline{\omega}^{\hat{\beta}} \tag{7.10}$$

we find the components of Riemann's curvature tensor.

6. Contraction gives the components of Ricci's curvature tensor

$$R_{\hat{\mu}\hat{\nu}} \equiv R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}\hat{\nu}} \tag{7.11}$$

7. A new contraction gives Ricci's curvature scalar

$$R \equiv R^{\hat{\mu}}_{\ \hat{\mu}} \tag{7.12}$$

8. The components of the Einstein tensor can then be found

$$E_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\eta_{\hat{\mu}\hat{\nu}}R \ , \tag{7.13}$$

where $\eta_{\hat{\mu}\hat{\nu}} = diag(-1, 1, 1, 1)$. We then have:

$$\begin{aligned} E_{\hat{t}\hat{t}} &= \frac{2}{r}e^{-2\beta}\beta' + \frac{1}{r^2}(1 - e^{-2\beta}) \\ E_{\hat{r}\hat{r}} &= \frac{2}{r}e^{-2\beta}\alpha' - \frac{1}{r^2}(1 - e^{-2\beta}) \\ E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} &= e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{\alpha'}{r} - \frac{\beta'}{r}) \end{aligned} \tag{7.14}$$

We want to solve the equations $E_{\hat{\mu}\hat{\nu}} = 0$. We get only 2 independent equations, and choose to solve those:

$$E_{\hat{t}\hat{t}} = 0 \qquad \text{and} \qquad E_{\hat{r}\hat{r}} = 0 \tag{7.15}$$

By adding the 2 equations we get:

$$E_{\hat{t}\hat{t}} + E_{\hat{r}\hat{r}} = 0$$

$$\Rightarrow \frac{2}{r}e^{-2\beta}(\beta' + \alpha') = 0$$

$$\Rightarrow (\alpha + \beta)' = 0 \Rightarrow \alpha + \beta = K_1 \quad \text{(const)} \tag{7.16}$$

We now have:

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2 \tag{7.17}$$

By choosing a suitable coordinate time, we can achieve

$$K_1 = 0 \Rightarrow \alpha = -\beta$$

Since we have $ds^2 = -e^{2\alpha}dt^2 + e^{-2\alpha}dr^2 + r^2 d\Omega^2$, this means that $g_{rr} = -\frac{1}{g_{tt}}$. We still have to solve one more equation to get the complete solution, and choose the equation $E_{\hat{t}\hat{t}} = 0$, which gives

$$\frac{2}{r}e^{-2\beta}\beta' + \frac{1}{r^2}(1 - e^{-2\beta}) = 0$$

This equation can be written:

$$\frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\beta})] = 0$$
$$\therefore r(1-e^{-2\beta}) = K_2 \qquad \text{(const)} \tag{7.18}$$

If we choose $K_2 = 0$ we get $\beta = 0$ giving $\alpha = 0$ and

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \ , \tag{7.19}$$

which is the Minkowski space-time described in spherical coordinates. In general, $K_2 \neq 0$ and $1 - e^{-2\beta} = \frac{K_2}{r} \equiv \frac{K}{r}$, giving

$$e^{2\alpha} = e^{-2\beta} = 1 - \frac{K}{r}$$

and

$$ds^2 = -(1 - \frac{K}{r})dt^2 + \frac{dr^2}{1-\frac{K}{r}} + r^2 d\Omega^2 \tag{7.20}$$