Lecture 31. 14. may 2018

Clocks at rest in the Boyer-Lindquist coordinate system show a proper time given by

$$d\tau = \sqrt{-g_{tt}}\,dt = \sqrt{1-\frac{2Mr}{\rho^2}}\,dt\ ,\ \rho^2 = r^2 + a^2\cos^2\theta\ .$$

Light emitted from the surface, $r = r_0$, where $g_{tt} = 0$ is infinitely redshifted further out. Observed from the outside time stands still.

$$\begin{aligned} \rho^2 = 2Mr_0 \quad &\Rightarrow \quad r_0^2 + a^2\cos^2\theta = 2Mr_0 \\ &r_0 = M \pm \sqrt{M^2 - a^2\cos^2\theta} \end{aligned} \tag{8.17}$$

This is the equation for the surface which represents infinite redshift.

To be accurate: Light emitted by sources at rest on the surface $r = r_0$ is infinitely redshifted as observed by outside observers.

To have stationary orbits the following must be true

$$g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt} < 0$$

This implies that Ω must be in the interval

$$\Omega_{min} < \Omega < \Omega_{max}\ ,$$

where $\Omega_{min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$, $\Omega_{max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$ since $g_{t\phi} = -\omega g_{\phi\phi}$.

Outside the surface with infinite redshift $g_{tt} < 0$. That is Ω can be negative, zero and positive. Inside the surface $r = r_0$ with infinite redshift $g_{tt} > 0$. Here $\Omega_{min} > 0$ and static particles, $\Omega = 0$, cannot exist. This is due to the inertial dragging effect. The surface $r = r_0$ is therefore known as "the static border".

The interval of Ω, where stationary orbits are allowed, is reduced to zero when $\Omega_{min} = \Omega_{max}$, that is $\omega^2 = \frac{g_{tt}}{g_{\phi\phi}} \Rightarrow g_{tt} = \omega^2 g_{\phi\phi}$ (equation for the horizon).

For the Kerr metric we have:

$$g_{tt} = \omega^2 g_{\phi\phi} - e^{2\nu} \tag{8.27}$$

Therefore the horizon equation becomes

$$e^{2\nu} = 0 \quad \Rightarrow \Delta = 0 \qquad \therefore r^2 - 2mr + a^2 = 0 \tag{8.28}$$

The largest solution is $r_+ = M + \sqrt{M^2 - a^2}$ and this is the equation for a spherical surface. The static border is $r_0 = M + \sqrt{M^2 - a^2\cos\theta}$.

Going from a region outside the static border and inwards we have the following situation. In the outside region it is possible for an observer to stay at rest if a suitable non-gravitational force acts upon the observer. The surface $r = r_0$ has two roles; light emitted from sources at rest on it are infinitely redshifted, and inside the surface the inertial dragging is so strong that it is impossible to stay at rest even by trying to do so by means of a motor. But it is possible to move along a path with constant r, i.e. there exist stationary orbits. Also it is possible to come out of this region. But inside the horizon, everything fall inwards, and it is not possible neither to move with constant *r* nor to move outwards.

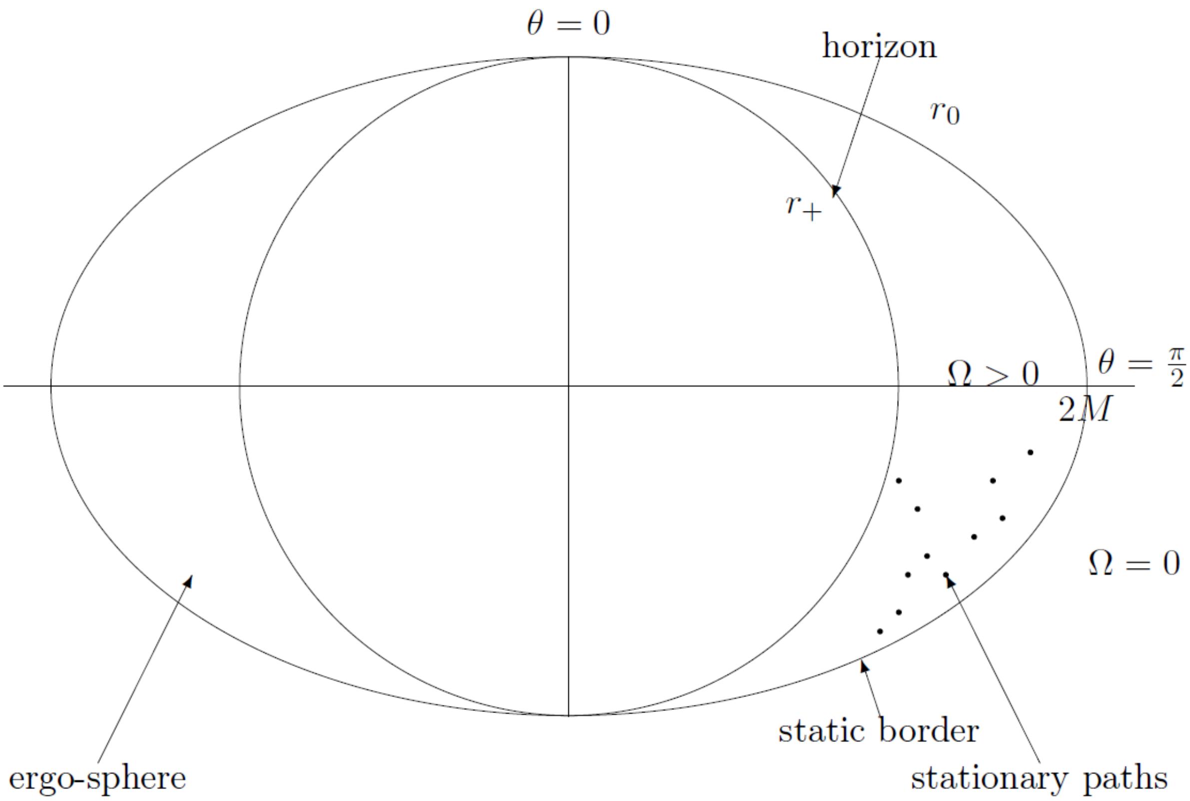

Figure 8.1: Static border and horizon of a Kerr black hole

9.2 The pressure contribution to the gravitational mass of a static, spherical symmetric system

We now give a new definition of the gravitational acceleration (not equivalent to (7.23))

$$g = +\frac{a}{u^t}\ , \qquad a = \sqrt{a_\mu a^\mu} \tag{9.10}$$

We have the line element:

$$\begin{aligned} ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \\ g_{tt} = -e^{2\alpha}\ , \quad g_{rr} = e^{2\beta} \end{aligned} \tag{9.11}$$

gives (because of the gravitational acceleration)

$$g = +e^{\alpha-\beta}\alpha' \tag{9.12}$$

From the expressions for $E_{\hat{t}\hat{t}}$, $E_{\hat{r}\hat{r}}$, $E_{\hat{\theta}\hat{\theta}}$, $E_{\hat{\phi}\hat{\phi}}$ follow (see Section 7.1)

$$E^{\hat{t}}_{\ \hat{t}} - E^{\hat{r}}_{\ \hat{r}} - E^{\hat{\theta}}_{\ \hat{\theta}} - E^{\hat{\phi}}_{\ \hat{\phi}} = -2e^{2\beta}(\frac{2\alpha'}{r} + \alpha'' + \alpha'^2 - \alpha'\beta')\ . \tag{9.13}$$

We also have

$$(r^2 e^{\alpha-\beta}\alpha')' = r^2 e^{\alpha-\beta}(\frac{2\alpha'}{r} + \alpha'' + \alpha'^2 - \alpha'\beta')\ , \tag{9.14}$$

which gives

$$g = +\frac{1}{2r^2}\int (E^{\hat{t}}_{\ \hat{t}} - E^{\hat{r}}_{\ \hat{r}} - E^{\hat{\theta}}_{\ \hat{\theta}} - E^{\hat{\phi}}_{\ \hat{\phi}}) r^2 e^{\alpha+\beta} dr\ . \tag{9.15}$$

By applying Einstein's field equations

$$E^{\hat{\mu}}_{\ \hat{\nu}} = 8\pi G T^{\hat{\mu}}_{\ \hat{\nu}} \tag{9.16}$$

we get

$$g = +\frac{4\pi G}{r^2}\int (T^{\hat{t}}_{\ \hat{t}} - T^{\hat{r}}_{\ \hat{r}} - T^{\hat{\theta}}_{\ \hat{\theta}} - T^{\hat{\phi}}_{\ \hat{\phi}}) r^2 e^{\alpha+\beta} dr\ . \tag{9.17}$$

This is the Tolman-Whittaker expression for gravitational acceleration.

The corresponding Newtonian expression is :

$$g_N = -\frac{4\pi G}{r^2}\int \rho r^2 dr \tag{9.18}$$

The relativistic gravitational mass density is therefore defined as

$$\rho_G = -T^{\hat{t}}_{\ \hat{t}} + T^{\hat{r}}_{\ \hat{r}} + T^{\hat{\theta}}_{\ \hat{\theta}} + T^{\hat{\phi}}_{\ \hat{\phi}} \tag{9.19}$$

For an isotropic fluid with

$$T^{\hat{t}}_{\hat{t}} = -\rho \ , \quad T^{\hat{r}}_{\hat{r}} = T^{\hat{\theta}}_{\hat{\theta}} = T^{\hat{\phi}}_{\hat{\phi}} = p \tag{9.20}$$

we get $\rho_G = \rho + 3p$ (with $c = 1$), which becomes

$$\boxed{\rho_G = \rho + \frac{3p}{c^2}} \tag{9.21}$$

It follows that in relativity, pressure has a gravitational effect. Greater pressure gives increasing gravitational attraction. Strain ($p < 0$) decreases the gravitational attraction.

In the Newtonian limit, $c \to \infty$, pressure has no gravitational effect.

Lorentz Invariant Vacuum Energy – LIVE

Let us follow a thought presented by the Belgian cosmologist Georges Lemaître around 1935. Assume a particle is alone in the universe. It is not possible to define motion for such a particle. Hence all motion is relative.

Quantum mechanics implies that particle-antiparticle pairs are created and then annihilated again in a very short time restricted by the Heisenberg uncertainty relationships. Averaging over a macroscopic time and region in space this implies the existence of a quantum mechanical vacuum energy on a macroscopic scale. If it is possible to measure velocity relative to this energy it would act as a sort of ether and re-establish absolute motion into the physics.

According to the special theory of relativity, which has been experimentally confirmed in several ways this cannot be the case. Hence it must be impossible to measure velocity relative to the vacuum energy. This implies that all the components of the energy-momentum tensor of the vacuum energy must be Lorentz invariant. One can show (See below) that Lorentz invariance of all the components of an energy-momentum tensor implies that the energy-momentum tensor is proportional to the metric tensor.

Assume now that the vacuum energy can be described as a perfect fluid.

$$T_{\mu\nu} = \left(\rho + p/c^2\right) u_\mu u_\nu + \rho g_{\mu\nu}$$

Lorentz invariance then requires that the vacuum energy obeys the equation of state $p = -\rho c^2$. Inserting this into eq.(9.21) gives $\rho_G = -2\rho < 0$. Hence *the assumption that it is not possible to measure velocity relative to the vacuum energy, and that the vacuum energy can be described as a perfect fluid, together with the general theory of relativity, imply that the vacuum energy produces repulsive gravitation.*

The energy-momentum tensor of a Lorentz invariant medium

All properties of the vacuum are Lorentz invariant. This means that the physical components $T_{\hat{\mu}\hat{\nu}}$ (i.e., the components in a local orthonormal basis $\{\mathbf{e}_{\hat{\mu}}\}$) of the energy-momentum density tensor for the vacuum, must have Lorentz invariant values. This must be valid for arbitrary Lorentz transformations $\Lambda^{\hat{\mu}}_{\hat{\mu}'}$. Thus

$$T_{\hat{\mu}\hat{\nu}} = T_{\hat{\mu}'\hat{\nu}'} = \Lambda^{\hat{\alpha}}_{\hat{\mu}'}\Lambda^{\hat{\beta}}_{\hat{\nu}'}T_{\hat{\alpha}\hat{\beta}} \,. \tag{A1}$$

Consider first a boost in the $x^{\hat{1}}$ direction

$$\Lambda^{\hat{\mu}}_{\hat{\mu}'} = \begin{pmatrix} \gamma & \gamma^{v} & 0 & 0 \\ \gamma^{v} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = (1 - v^2)^{-1/2} \,. \tag{A2}$$

Equations (A1) and (A2) give

$$T_{\hat{0}\hat{0}} = T_{\hat{0}'\hat{0}'} = \gamma^2 T_{\hat{0}\hat{0}} + \gamma^2 v^2 T_{\hat{1}\hat{1}} + \gamma^2 v (T_{\hat{0}\hat{1}} + T_{\hat{1}\hat{0}}) \,. \tag{A3}$$

Using that $\gamma^2 - 1 = \gamma^2 v^2$ Eq. (A3) can be written as

$$v = (T_{\hat{0}\hat{0}} + T_{\hat{1}\hat{1}}) + T_{\hat{0}\hat{1}} + T_{\hat{1}\hat{0}} = 0 \,. \tag{A4}$$

Transformation of $T_{\hat{1}\hat{1}}$ gives the same equation. In a similar way transformation of $T_{\hat{0}\hat{1}}$ and $T_{\hat{1}\hat{0}}$ leads to

$$T_{\hat{0}\hat{0}} + T_{\hat{1}\hat{1}} + v(T_{\hat{0}\hat{1}} + T_{\hat{1}\hat{0}}) = 0 \,. \tag{A5}$$

From Eqs. (A4) and (A5) follows

$$T_{\hat{1}\hat{1}} = -T_{\hat{0}\hat{0}}, \quad T_{\hat{0}\hat{1}} = -T_{\hat{1}\hat{0}} \,. \tag{A6}$$

Transformations of $T_{\hat{0}\hat{2}}$ and $T_{\hat{1}\hat{2}}$ give, respectively,

$$T_{\hat{0}\hat{2}} = \gamma T_{\hat{0}\hat{2}} + \gamma v T_{\hat{1}\hat{2}} \,, \tag{A7}$$

$$T_{\hat{1}\hat{2}} = \gamma v T_{\hat{0}\hat{2}} + \gamma T_{\hat{1}\hat{2}} \,, \tag{A8}$$

which demands that

$$T_{\hat{0}\hat{2}} = T_{\hat{1}\hat{2}} = 0 \,. \tag{A9}$$

In the same way one finds

$$T_{\hat{2}\hat{0}} = T_{\hat{2}\hat{1}} = T_{\hat{0}\hat{3}} = T_{\hat{1}\hat{3}} = T_{\hat{3}\hat{0}} = T_{\hat{3}\hat{1}} = 0 \,. \tag{A10}$$

Furthermore,

$$T_{\hat{2}\hat{2}} = \Lambda^{\hat{\alpha}}_{\hat{2}}\Lambda^{\hat{\beta}}_{\hat{2}}T_{\hat{\alpha}\hat{\beta}} = T_{\hat{2}\hat{2}} \,, \tag{A11}$$

which does not restrict $T_{\hat{2}\hat{2}}$. The transformation (A2) does not imply any restriction on $T_{\hat{3}\hat{3}}$, $T_{\hat{2}\hat{3}}$, or $T_{\hat{3}\hat{2}}$ either.

Thus as a result of Lorentz invariance of the components $T_{\hat{\mu}\hat{\nu}}$ under a boost in the $x^{\hat{1}}$ direction, we have obtained the following form of the energy-momentum density tensor for the vacuum:

$$T_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} T_{\hat{0}\hat{0}} & T_{\hat{0}\hat{1}} & 0 & 0 \\ -T_{\hat{0}\hat{1}} & -T_{\hat{0}\hat{0}} & 0 & 0 \\ 0 & 0 & T_{\hat{2}\hat{2}} & T_{\hat{2}\hat{3}} \\ 0 & 0 & T_{\hat{3}\hat{2}} & T_{\hat{3}\hat{3}} \end{pmatrix} . \tag{A12}$$

Demanding Lorentz invariance under a boost in the x^2 direction gives the additional equations

$$T_{\hat{0}\hat{1}} = T_{\hat{1}\hat{0}} = T_{\hat{2}\hat{3}} = T_{\hat{3}\hat{2}} = 0, \quad T_{\hat{2}\hat{2}} = -T_{\hat{0}\hat{0}} \,. \tag{A13}$$

At last, Lorentz invariance under a boost in the x^3 direction gives the additional equation

$$T_{\hat{3}\hat{3}} = -T_{\hat{0}\hat{0}} \,. \tag{A14}$$

From Eqs. (A12)–(A14) follows

$$T_{\hat{\mu}\hat{\nu}} = T_{\hat{0}\hat{0}}\ \mathrm{diag}(1, -1, -1, -1) \,, \tag{A15}$$

which can be written as

$$T_{\hat{\mu}\hat{\nu}} = T_{\hat{0}\hat{0}}\ \eta_{\hat{\mu}\hat{\nu}} \,, \tag{A16}$$

where $\eta_{\hat{\mu}\hat{\nu}}$ are the components of the Minkowski metric tensor. Transformation to an arbitrary basis $\{\mathbf{e}_\mu\}$ gives

$$T_{\mu\nu} = T_{\hat{0}\hat{0}}\ g_{\mu\nu} \,. \tag{A17}$$

From the physical interpretation of the components of the energy-momentum density tensor, it follows that

$$T_{\hat{0}\hat{0}} = \rho \,, \tag{A18}$$

where ρ is the energy density of the system (here the vacuum). This will not be invariant under Lorentz transformations for arbitrary systems. But, the energy density *of the vacuum* is in general a scalar function of the four space-time coordinates.

In homogeneous cosmological models one demands that the density ρ measured by an observer depends only upon time. Due to the relativity of simultaneity this condition is Lorentz invariant only if $\rho = \text{constant} = \rho_0$. Thus in the cosmological case the energy-momentum density tensor of the vacuum is

$$T_{\mu\nu} = \rho_0\, g_{\mu\nu} \,. \tag{A19}$$

The energy density of the vacuum appears as a cosmological constant.