Lecture 32. 15. May 2018

9.3 The Tolman-Oppenheimer-Volkov equation

With spherical symmetry the spacetime line-element may be written

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$
$$E_{\hat{t}\hat{t}} = 8\pi G T_{\hat{t}\hat{t}}\ , \quad T^{\hat{\mu}}_{\hat{\nu}} = diag(-\rho, p, p, p) \tag{9.22}$$

From $E_{\hat{t}\hat{t}}$ we get

$$\frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\beta})] = 8\pi G\rho$$
$$r(1-e^{-2\beta}) = 2G\int_0^r 4\pi\rho r^2 dr\ , \tag{9.23}$$

where $m(r) = \int_0^r 4\pi\rho r^2 dr$ giving

$$e^{-2\beta} = 1 - \frac{2Gm(r)}{r} = \frac{1}{g_{rr}} \tag{9.24}$$

From $E_{\hat{r}\hat{r}}$ we have

$$E_{\hat{r}\hat{r}} = 8\pi G T_{\hat{r}\hat{r}}$$
$$\frac{2}{r}\frac{d\alpha}{dr}e^{-2\beta} - \frac{1}{r^2}(1-e^{-2\beta}) = 8\pi G p \tag{9.25}$$

We get

$$\frac{2}{r}\frac{d\alpha}{dr}(1-\frac{2Gm(r)}{r}) - \frac{2Gm(r)}{r^3} = 8\pi G p$$

$$\frac{d\alpha}{dr} = G\frac{m(r)+4\pi r^3 p(r)}{r(r-2Gm(r))} \tag{9.26}$$

The relativistic equation of hydrostatic equilibrium is $T^{\hat{r}\hat{\nu}}{}_{;\nu} = 0$. Written out

$$T^{\hat{r}\hat{\nu}}{}_{,\hat{\nu}} + \Gamma^{\nu}{}_{\hat{\alpha}\hat{\nu}}T^{\hat{r}\hat{\alpha}} + \Gamma^{\hat{r}}{}_{\hat{\alpha}\hat{\nu}}T^{\hat{\alpha}\hat{\nu}} = 0\ .$$

The first term is

$$T^{\hat{r}\hat{\nu}}{}_{,\hat{\nu}} = T^{\hat{r}\hat{r}}{}_{,\hat{r}} = p_{,\hat{r}} = \vec{e}_{\hat{r}}(p)\ .$$

Since $\vec{e}_r \cdot \vec{e}_r = g_{rr}$ we have $|\vec{e}_r| = \sqrt{g_{rr}}$. Hence, $\vec{e}_{\hat{r}} = \left(1/\sqrt{g_{rr}}\right)\vec{e}_r$. This gives

$$T^{\hat{r}\hat{v}}{}_{,\hat{v}} = \vec{e}_{\hat{r}}(p) = \frac{1}{\sqrt{g_{rr}}}\vec{e}_r(p) = e^{-\beta}\frac{dp}{dr} .$$

We here use ordinary derivatives instead of partial derivatives since *p* only depends upon *r*. The second term is

$$\Gamma^{\hat{v}}{}_{\hat{\alpha}\hat{v}}T^{\hat{r}\hat{\alpha}} = \Gamma^{\hat{v}}{}_{\hat{r}\hat{v}}T^{\hat{r}\hat{r}} = \Gamma^{\hat{v}}{}_{\hat{r}\hat{v}}p = \Gamma^{\hat{t}}{}_{\hat{r}\hat{t}}p + \Gamma^{\hat{i}}{}_{\hat{r}\hat{i}}p .$$

The third term is

$$\Gamma^{\hat{r}}{}_{\hat{\alpha}\hat{v}}T^{\hat{\alpha}\hat{v}} = \Gamma^{\hat{r}}{}_{\hat{v}\hat{v}}T^{\hat{v}\hat{v}} = \Gamma^{\hat{r}}{}_{\hat{t}\hat{t}}\rho + \Gamma^{\hat{r}}{}_{\hat{i}\hat{i}}p .$$

In orthonormal basis we have

$$\Omega_{\hat{v}\hat{\mu}} = -\Omega_{\hat{\mu}\hat{v}} \quad \Rightarrow \quad \Gamma_{\hat{v}\hat{\mu}\hat{\alpha}} = \Gamma_{\hat{\mu}\hat{v}\hat{\alpha}} \qquad \text{and} \qquad \Gamma^{\hat{i}}{}_{\hat{r}\hat{i}} = \Gamma_{\hat{i}\hat{r}\hat{i}} = -\Gamma_{\hat{r}\hat{i}\hat{i}} = -\Gamma^{\hat{r}}{}_{\hat{i}\hat{i}} .$$

Hence the last terms in the expressions for $\Gamma^{\hat{v}}{}_{\hat{\alpha}\hat{v}}T^{\hat{r}\hat{\alpha}}$ and $\Gamma^{\hat{r}}{}_{\hat{\alpha}\hat{v}}T^{\hat{\alpha}\hat{v}}$ cancel each other, and the hydrostatic equation, $T^{\hat{r}\hat{v}}{}_{;v} = 0$, then takes the form

$$e^{-\beta}\frac{dp}{dr} + \Gamma^{\hat{t}}{}_{\hat{r}\hat{t}}p + \Gamma^{\hat{r}}{}_{\hat{t}\hat{t}}\rho = 0 . \qquad (9.29)$$

Furthermore

$$\Gamma^{\hat{t}}{}_{\hat{r}\hat{t}} = -\Gamma_{\hat{t}\hat{r}\hat{t}} = \Gamma_{\hat{r}\hat{t}\hat{t}} = \Gamma^{\hat{r}}{}_{\hat{t}\hat{t}} ,$$

and

$$\Gamma^{\hat{r}}{}_{\hat{t}\hat{t}} = e^{-\beta}\frac{d\alpha}{dr} .$$

The hydrostatic equation then takes the form

$$\frac{dp}{dr} + (p+\rho)\frac{d\alpha}{dr} = 0 . \qquad (9.31)$$

Inserting Equation 9.26 into Equation 9.31 gives

$$\boxed{\frac{dp}{dr} = -G(\rho + p)\frac{m(r) + 4\pi r^3 p(r)}{r(r - 2Gm(r))}} \qquad (9.32)$$

This is the Tolman-Oppenheimer-Volkov (TOV) equation which can be used to construct relativistic star models.

The metric component $g_{tt} = -e^{2\alpha(r)}$ can now be calculated from eq.(9.31) in the form

$$d\alpha = -\frac{dp}{\rho + p}. \tag{9.33}$$

9.4 An exact solution for incompressible stars - Schwarzschild's interior solution

We now consider an incompressible star, say a neutron star, with $\rho =$ constant and radius *R*. Integration of eq.(9.33) with vanishing pressure at the surface then gives

$$\int_{\alpha(R)}^{\alpha(r)} d\alpha = -\int_0^p \frac{dp}{\rho + p} \Rightarrow \alpha(r) = \alpha(R) - \ln\left(1 + \frac{p}{\rho}\right).$$

Requiring continuity of the metric at $r = R$, and using the exterior Schwarzschild metric, we get

$$g_{tt}(r) = e^{2\alpha(r)} = \frac{g_{tt}(R)}{\left(1 + \frac{p}{\rho}\right)^2} = -\frac{1 - \frac{R_S}{R}}{\left(1 + \frac{p}{\rho}\right)^2}. \tag{9.34}$$

For an incompressible star the mass inside a radius *r* is

$$m(r) = \frac{4\pi}{3}\rho r^3.$$

We then get

$$e^{-2\beta} = 1 - \frac{2Gm(r)}{r} = 1 - \frac{8\pi G\rho}{3} r^2.$$

Defining a constant *a* with dimension length by

$$a^2 = \frac{3}{8\pi G\rho},$$

this may be written as

$$e^{-2\beta} = 1 - \frac{r^2}{a^2}.$$

It may be noted that

$$R_S = 2Gm(R) = \frac{8\pi G\rho}{3}R^3 = \frac{R^3}{a^2}. \quad \text{Hence} \quad a^2 = \frac{R^3}{R_S} \quad \text{and} \quad \frac{R^2}{a^2} = \frac{R_S}{R}.$$

The TOV-equation may now be written

$$\frac{dp}{dr} = -\frac{1}{2a^2\rho}(\rho + 3p)(\rho + p)\frac{r}{1 - \frac{r^2}{a^2}} .$$

Hence

$$\int_0^p \frac{dp}{(\rho + 3p)(\rho + p)} = -\frac{1}{2a^2\rho}\int_R^r \frac{r}{1 - \frac{r^2}{a^2}}dr ,$$

which gives

$$\frac{\rho + p}{\rho + 3p} = \sqrt{\frac{a^2 - R^2}{a^2 - r^2}} ,$$

giving the pressure distribution

$$p(r) = \frac{\sqrt{a^2 - r^2} - \sqrt{a^2 - R^2}}{3\sqrt{a^2 - R^2} - \sqrt{a^2 - r^2}}\rho = \frac{\sqrt{1 - \frac{R_s}{R^3}r^2} - \sqrt{1 - \frac{R_S}{R}}}{3\sqrt{1 - \frac{R_S}{R}} - \sqrt{1 - \frac{R_s}{R^3}r^2}}\rho \quad , \quad r < R .$$

This gives

$$1 + \frac{p}{\rho} = \frac{2\sqrt{1 - \frac{R_S}{R}}}{3\sqrt{1 - \frac{R_S}{R}} - \sqrt{1 - \frac{R_s}{R^3}r^2}} .$$

Combining this with equation (9.34) we get

$$g_{tt} = -\frac{1}{4}\left(3\sqrt{1 - \frac{R_S}{R}} - \sqrt{1 - \frac{R_S}{R^3}r^2}\right)^2 .$$

Hence the line-element of the space-time inside an incompressible star is

$$ds^2 = -\frac{1}{4}\left(3\sqrt{1 - \frac{R_S}{R}} - \sqrt{1 - \frac{R_S}{R^3}r^2}\right)^2 dt^2 + \frac{dr^2}{1 - \frac{R_S}{R^3}r^2} + r^2 d\Omega^2 .$$

The spacetime described by this line-element is called the internal Schwarzschild space-time.

In Newtonian gravity there is no limit to how large a star can be if there exists a sufficiently effective mechanism for generating a pressure gradient which may resist gravity. We shall see that this is different in the case of relativistic gravity.

In order to satisfy the condition of hydrostatic equilibrium the pressure must be positive. This must be valid at every distance from the center of the star, and also at the center, $p(0) > 0$. It follows from the expression for the pressure distribution that

$$p(0) = \frac{1 - \sqrt{1 - \frac{R_S}{R}}}{3\sqrt{1 - \frac{R_S}{R}} - 1} > 0 ,$$

which requires that $R_S < (8/9)R$. Including here the velocity of light, this leads to the condition

$$M < \frac{c^2}{G\sqrt{3\pi G\rho}} = \frac{2\sqrt{2}}{3}\frac{c^2}{G}a , \qquad a = c\sqrt{\frac{3}{8\pi G\rho}} .$$

A star with a larger mass will collapse to a black hole independent of the mechanism that generates a pressure gradient in the star.

Let us consider an incompressible neutron star as an illustrating example. A typical density is then $\rho_n = 5 \cdot 10^{17}\,\mathrm{kg/m^3}$. This gives $a_n = 56.8\,\mathrm{km}$ and $M < 128 \cdot 10^{30}\,\mathrm{kg} = 64 M_{\odot}$.