2.1 Vectors

An expression on the form $a^\mu \vec{e}_\mu$, where a^μ, $\mu = 1, 2, ..., n$ are real numbers, is known as a **linear combination** of the vectors $\vec{e}_\mu$.

The vectors $\vec{e}_1, ..., \vec{e}_n$ are said to be linearly independent if there does **not** exist real numbers $a^\mu \neq 0$ such that $a^\mu \vec{e}_\mu = 0$.

Geometrical interpretation: A set of vectors are **linearly independent** if it is **not** possible to construct a closed polygon of the vectors (even by adjusting their lengths).

A set of vectors $\vec{e}_1, \ldots, \vec{e}_n$ are said to be **maximally linearly independent** if $\vec{e}_1, \ldots, \vec{e}_n, \vec{v}$ are linearly dependent for all vectors $\vec{v} \neq \vec{e}_\mu$. We define the **dimension** of a vector-space as the number of vectors in a maximally linearly independent set of vectors of the space. The vectors $\vec{e}_\mu$ in such a set are known as the **basis-vectors** of the space.

$$\begin{aligned} \vec{v} + a^\mu \vec{e}_\mu &= 0 \\ &\Downarrow \\ \vec{v} &= -a^\mu \vec{e}_\mu \end{aligned} \tag{2.1}$$

The components of $\vec{v}$ are the numbers v^μ defined by $v^\mu = -a^\mu \Rightarrow \vec{v} = v^\mu \vec{e}_\mu$.

2.1.1 4-vectors

4-vectors are vectors which exist in (4-dimensional) space-time. A 4-vector equation represents 4 independent component equations.

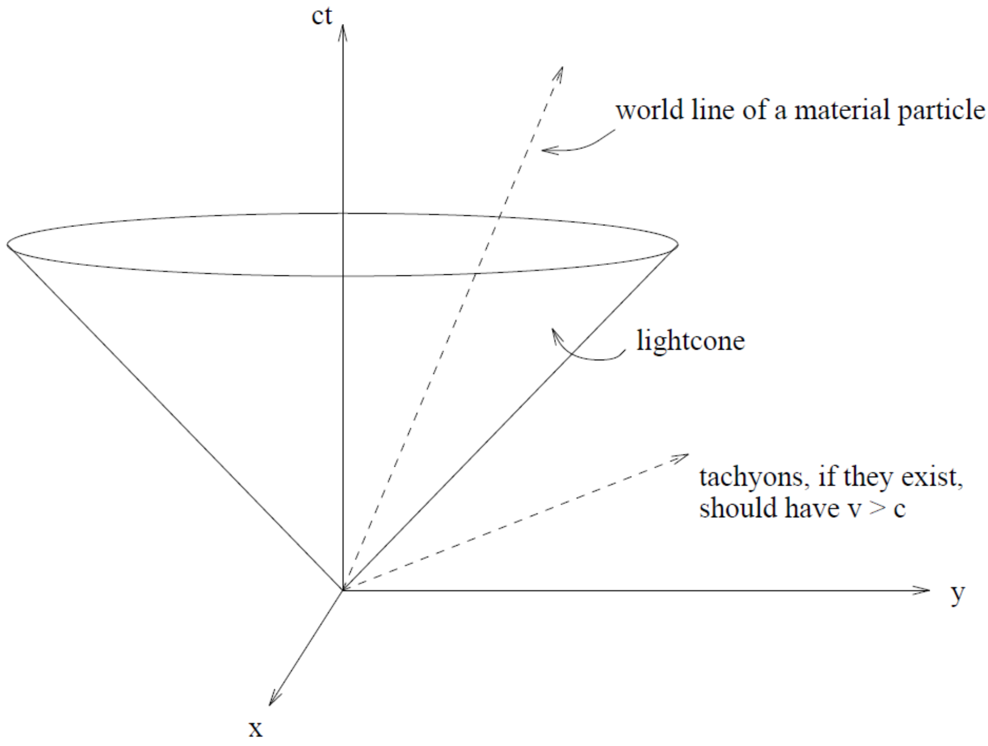

Figure 2.3: World-lines in a Minkowski diagram

The proper time-interval is denoted by $d\tau$ (above it was denoted Δt_0). The proper time-interval for a particle is measured with a standard clock which follows the particle.

Definition 2.1.1 (4-velocity)

$$\vec{U} = c\frac{dt}{d\tau}\vec{e}_t + \frac{dx}{d\tau}\vec{e}_x + \frac{dy}{d\tau}\vec{e}_y + \frac{dz}{d\tau}\vec{e}_z, \tag{2.3}$$

where t is the coordinate time, measured with clocks at rest in the reference frame.

$$\begin{aligned} \vec{U} &= U^\mu \vec{e}_\mu = \frac{dx^\mu}{d\tau}\vec{e}_\mu, \quad x^\mu = (ct, x, y, z), \quad x^0 \equiv ct \\ \frac{dt}{d\tau} &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma \end{aligned} \tag{2.4}$$

$\vec{U} = \gamma(c, \vec{v})$, where $\vec{v}$ is the common 3-velocity of the particle.

Definition 2.1.2 (4-momentum)

$$\vec{P} = m_0 \vec{U}, \tag{2.5}$$

where m_0 is the rest mass of the particle.

$\vec{P} = (\frac{E}{c}, \vec{p})$, where $\vec{p} = \gamma m_0 \vec{v} = m\vec{v}$ and E is the relativistic energy.

The 4-force or Minkowski-force $\vec{F} \equiv \frac{d\vec{P}}{d\tau}$ and the 'common force' $\vec{f} = \frac{d\vec{p}}{dt}$. Then

$$\vec{F} = \gamma(\frac{1}{c}\vec{f} \cdot \vec{v}, \vec{f}) \tag{2.6}$$

Definition 2.1.3 (4-acceleration)

$$\vec{A} = \frac{d\vec{U}}{d\tau} \tag{2.7}$$

The 4-velocity has the scalar value c so that

$$\vec{U} \cdot \vec{U} = -c^2 \tag{2.8}$$

The *4-velocity identity* eq. 2.8 gives $\vec{U} \cdot \vec{A} = 0$, in other words $\vec{A} \perp \vec{U}$ and $\vec{A}$ is space-like.

The line element for Minkowski space-time (flat space-time) with Cartesian coordinates is

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.9}$$

In general relativity theory, gravitation is not considered a force. Gravitation is instead described as motion in a curved space-time.

A particle in free fall, is in Newtonian gravitational theory said to be only influenced by the gravitational force. According to general relativity theory the particle is not influenced by any force.

Such a particle has no 4-acceleration. $\vec{A} \neq 0$ implies that the particle is not in free fall. It is then influenced by non-gravitational forces.

One has to distinguish between observed acceleration, ie. common 3-acceleration, and the absolute 4-acceleration.

Definition 2.1.4 (Reference frame)
A **reference frame** is defined as a continuum of non-intersecting timelike world lines in spacetime.

We can view a reference frame as a set of reference particles with a specified motion. An *inertial reference frame* is a non-rotating set of free particles.

Definition 2.1.5 (Coordinate system)
A **coordinate system** is a continuum of 4-tuples giving a unique set of coordinates for events in spacetime.

Definition 2.1.6 (Comoving coordinate system)
A **comoving coordinate system** in a frame is a coordinate system where the particles in the reference frame have constant spatial coordinates.

Definition 2.1.7 (Orthonormal basis)
An **orthonormal basis** $\{\vec{e}_{\hat{\mu}}\}$ in spacetime is defined by

$$\vec{e}_{\hat{t}} \cdot \vec{e}_{\hat{t}} = -1 (c = 1)$$
$$\vec{e}_{\hat{i}} \cdot \vec{e}_{\hat{j}} = \delta_{\hat{i}\hat{j}}$$

where $\hat{i}$ and $\hat{j}$ are space indices.

Definition 2.1.8 (Coordinate basis vectors.)
Temporary definition of coordinate basis vector:
Assume any coordinate system $\{x^\mu\}$.

$$\vec{e}_\mu \equiv \frac{\partial \vec{r}}{\partial x^\mu}$$

A *vector field* is a continuum of vectors in a space, where the components are continuous and differentiable functions of the coordinates. Let $\vec{v}$ be a tangent vector to the curve $\vec{r}(\lambda)$:

$$\vec{v} = \frac{d\vec{r}}{d\lambda} \qquad \text{where} \quad \vec{r} = \vec{r}[x^\mu(\lambda)] \tag{2.12}$$

The chain rule for differentiation yields:

$$\vec{v} = \frac{d\vec{r}}{d\lambda} = \frac{\partial \vec{r}}{\partial x^{\mu}}\frac{dx^{\mu}}{d\lambda} = \frac{dx^{\mu}}{d\lambda}\vec{e_{\mu}} = v^{\mu}\vec{e}_{\mu} \tag{2.13}$$

Thus, the components of the tangent vector field along a curve, parameterised by λ, is given by:

$$v^{\mu} = \frac{dx^{\mu}}{d\lambda} \tag{2.14}$$

In the theory of relativity, the invariant parameter is often chosen to be the proper time. Tangent vector to the world line of a material particle:

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} \tag{2.15}$$

These are the components of the 4-velocity of the particle!

Digression 2.1.1 (Proper time of the photon.)
Minkowski-space:

$$\begin{aligned} ds^2 &= -c^2dt^2 + dx^2 \\ &= -c^2dt^2(1 - \frac{1}{c^2}(\frac{dx}{dt})^2) \\ &= -(1 - \frac{v^2}{c^2})c^2dt^2 \end{aligned} \tag{2.16}$$

For a photon, $v = c$ so:

$$\lim_{v \to c} ds^2 = 0 \tag{2.17}$$

Thus, the spacetime interval between two points on the world line of a photon, is zero! This also means that the proper time for the photon is zero!! (See example 2.1.2).

Digression 2.1.2 (Relationships between spacetime intervals, time and proper time.)
Physical interpretation of the spacetime interval for a timelike interval:

$$ds^2 = -c^2d\tau^2 \tag{2.18}$$

where dτ is the proper time interval between two events, measured on a clock moving in a way, such that it is present on both events (figure 2.6).

$$\begin{aligned} -c^2d\tau^2 &= -c^2(1 - \frac{v^2}{c^2})dt^2 \\ \Rightarrow d\tau &= \sqrt{1 - \frac{v^2}{c^2}}dt \end{aligned} \tag{2.19}$$

The time interval between to events in the laboratory, is smaller measured on a moving clock than measured on a stationary one, because the moving clock is ticking slower!