Lecture 5 29.01.2018

2.1.3 Coordinate transformations

Given two coordinate systems $\{x^\mu\}$ and $\{x^{\mu'}\}$.

$$\vec{e}_{\mu'} = \frac{\partial \vec{r}}{\partial x^{\mu'}} \tag{2.20}$$

Suppose there exists a coordinate transformation, such that the primed coordinates are functions of the unprimed, and vice versa. Then we can apply the chain rule:

$$\vec{e}_{\mu'} = \frac{\partial \vec{r}}{\partial x^{\mu'}} = \frac{\partial \vec{r}}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} = \vec{e}_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} \tag{2.21}$$

This is the transformation equation for the basis vectors. $\frac{\partial x^\mu}{\partial x^{\mu'}}$ are elements of the transformation matrix. Indices that are *not* sum-indices are called 'free indices'.

> **Rule:** In *all terms* on each side in an equation, the free indices should behave identically (high or low), **and** there should be exactly the *same* indices in all terms!

Applying this rule, we can now find the inverse transformation

$$\begin{aligned}
\vec{e}_\mu &= \vec{e}_{\mu'}\frac{\partial x^{\mu'}}{\partial x^\mu} \\
\vec{v} &= v^{\mu'}\vec{e}_{\mu'} = v^\mu \vec{e}_\mu = v^{\mu'}\vec{e}_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}
\end{aligned}$$

So, the transformation rules for the *components* of a vector becomes

$$\boxed{v^\mu = v^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\,; \qquad v^{\mu'} = v^\mu \frac{\partial x^{\mu'}}{\partial x^\mu}} \tag{2.22}$$

The directional derivative along a curve, parametrised by λ:

$$\frac{d}{d\lambda} = \frac{\partial}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = v^\mu \frac{\partial}{\partial x^\mu} \tag{2.23}$$

where $v^\mu = \frac{dx^\mu}{d\lambda}$ are the components of the tangent vector of the curve. Directional derivative along a coordinate curve:

$$\lambda = x^\nu \qquad \frac{\partial}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu} \tag{2.24}$$

In the primed system:

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu} \tag{2.25}$$

Definition 2.1.9 (Coordinate basis vectors.)
We define the coordinate basis vectors as:

$$\boxed{\vec{e}_\mu = \frac{\partial}{\partial x^\mu}} \tag{2.26}$$

This definition is not based upon the existence of finite position vectors. It applies in curved spaces as well as in flat spaces.

Example 2.1.2 (Coordinate transformation)
From Figure 2.7 we see that

$$\boxed{x = r\cos\theta,\ y = r\sin\theta}$$

Coordinate basis vectors were defined by

$$\vec{e_\mu} \equiv \frac{\partial}{\partial x^\mu}$$

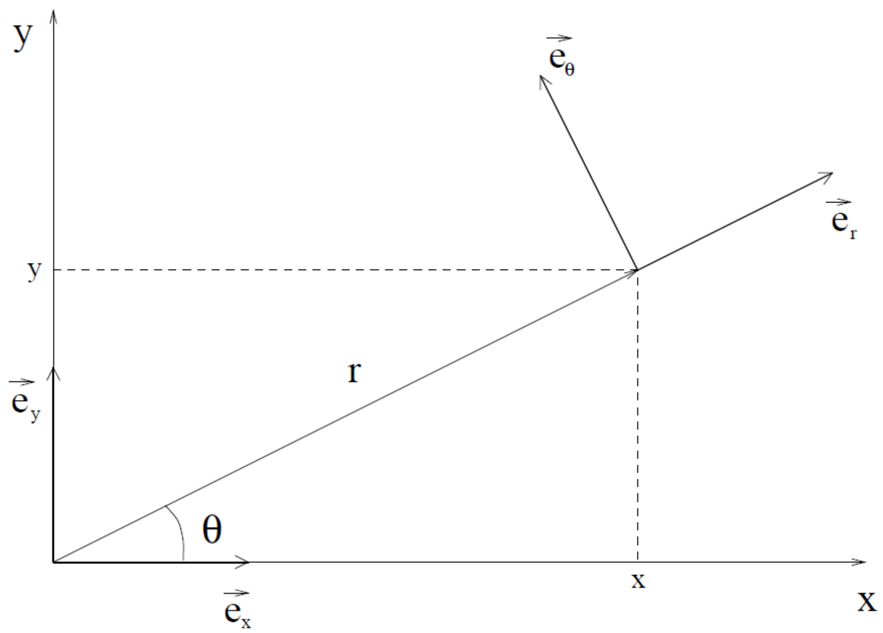

Figure 2.7: Coordinate transformation, flat space.

This means that we have

$$\vec{e_x} = \frac{\partial}{\partial x} \,, \quad \vec{e_y} = \frac{\partial}{\partial y} \,, \quad \vec{e_r} = \frac{\partial}{\partial r} \,, \quad \vec{e_\theta} = \frac{\partial}{\partial \theta}$$
$$\vec{e_r} = \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \tag{2.29}$$

Using the chain rule and Equations (2.27) and (2.29) we get

$$\begin{aligned} \vec{e_r} &= \cos\theta \vec{e_x} + \sin\theta \vec{e_y} \\ \vec{e_\theta} &= \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} \\ &= -r\sin\theta \vec{e_x} + r\cos\theta \vec{e_y} \end{aligned} \tag{2.30}$$

But are the vectors in (2.30) also unit vectors?

$$\vec{e_r} \cdot \vec{e_r} = \cos^2\theta + \sin^2\theta = 1 \tag{2.31}$$

So $\vec{e_r}$ is a unit vector, $|\vec{e_r}| = 1$.

$$\vec{e_\theta} \cdot \vec{e_\theta} = r^2(\cos^2\theta + \sin^2\theta) = r^2 \tag{2.32}$$

and we see that $\vec{e_\theta}$ is **not** a unit vector, $|\vec{e_\theta}| = r$. But we have that $\vec{e_r} \cdot \vec{e_\theta} = 0 \Rightarrow \vec{e_r} \perp \vec{e_\theta}$. **Coordinate basis vectors are not generally unit vectors.**

Definition 2.1.10 (Orthonormal basis)
An orthonormal basis is a vector basis consisting of unit vectors that are normal to each other. To show that we are using an orthonormal basis we will use 'hats' over the indices, $\{\vec{e}_{\hat{\mu}}\}$.

Orthonormal basis associated with planar polar coordinates:

$$\vec{e}_{\hat{r}} = \vec{e}_r \,, \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta \tag{2.33}$$

2.1.4 Structure coefficients

Definition 2.1.11 (Commutators between vectors)
The commutator between two vectors, $\vec{u}$ and $\vec{v}$, is defined as

$$[\vec{u}\,,\,\vec{v}] \equiv \vec{u}\vec{v} - \vec{v}\vec{u} \tag{2.35}$$

where $\vec{u}\vec{v}$ is defined as

$$\vec{u}\vec{v} \equiv u^\mu \vec{e_\mu}(v^\nu \vec{e_\nu}) = u^\mu \frac{\partial}{\partial x^\mu}(v^\nu \frac{\partial}{\partial x^\nu}) \tag{2.36}$$

We can think of a vector as a linear combination of partial derivatives. We get:

$$\begin{aligned}\vec{u}\vec{v} &= u^\mu \frac{\partial v^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + u^\mu v^\nu \frac{\partial^2}{\partial x^\mu \partial x^\nu} \\ &= u^\mu \frac{\partial v^\nu}{\partial x^\mu}\vec{e_\nu} + u^\mu v^\nu \frac{\partial^2}{\partial x^\mu \partial x^\nu}\end{aligned} \tag{2.37}$$

Due to the last term, $\vec{u}\vec{v}$ is **not** a vector.

$$\begin{aligned}\vec{v}\vec{u} &= v^\nu \frac{\partial}{\partial x^\nu}(u^\mu \frac{\partial}{\partial x^\mu}) \\ &= v^\nu \frac{\partial u^\mu}{\partial x^\nu}\vec{e_\mu} + v^\nu u^\mu \frac{\partial^2}{\partial x^\nu \partial x^\mu} \\ \vec{u}\vec{v} - \vec{v}\vec{u} &= u^\mu \frac{\partial v^\nu}{\partial x^\mu}\vec{e_\nu} - \underbrace{v^\nu \frac{\partial u^\mu}{\partial x^\nu}\vec{e_\mu}}_{v^\mu \frac{\partial u^\nu}{\partial x^\mu}\vec{e_\nu}} \\ &= (u^\mu \frac{\partial v^\nu}{\partial x^\mu} - v^\mu \frac{\partial u^\nu}{\partial x^\mu})\vec{e_\nu}\end{aligned} \tag{2.38}$$

Here we have used that

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu} \tag{2.39}$$

The Einstein comma notation $\Rightarrow$

$$\vec{u}\vec{v} - \vec{v}\vec{u} = (u^\mu v^\nu{}_{,\mu} - v^\mu u^\nu{}_{,\mu})\vec{e_\nu} \tag{2.40}$$

As we can see, the commutator between two vectors is itself a vector.

Definition 2.1.12 (Structure coefficients $c^\rho{}_{\mu\nu}$)
The structure coefficients $c^\rho{}_{\mu\nu}$ in an arbitrary basis $\{\vec{e_\mu}\}$ are defined by:

$$[\vec{e_\mu}\ ,\ \vec{e_\nu}] \equiv c^\rho{}_{\mu\nu}\vec{e_\rho}$$

Structure coefficients in a coordinate basis:

$$\begin{aligned}[\vec{e_\mu}\ ,\ \vec{e_\nu}] &= [\frac{\partial}{\partial x^\mu}\ ,\ \frac{\partial}{\partial x^\nu}] \\ &= \frac{\partial}{\partial x^\mu}(\frac{\partial}{\partial x^\nu}) - \frac{\partial}{\partial x^\nu}(\frac{\partial}{\partial x^\mu}) \\ &= \frac{\partial^2}{\partial x^\mu \partial x^\nu} - \frac{\partial^2}{\partial x^\nu \partial x^\mu} = 0\end{aligned}$$

The commutator between two coordinate basis vectors is zero, so the structure coefficients are zero in coordinate basis.

Example 2.1.4 (Structure coefficients in planar polar coordinates)
We will find the structure coefficients of an orthonormal basis in planar polar coordinates. In (2.33) we found that

$$\vec{e}_{\hat{r}} = \vec{e}_r \ , \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta \tag{2.43}$$

We will now use this to find the structure coefficients.

$$\begin{aligned}[\vec{e}_{\hat{r}} \ , \ \vec{e}_{\hat{\theta}}] &= [\frac{\partial}{\partial r} \ , \ \frac{1}{r}\frac{\partial}{\partial \theta}] \\ &= \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial \theta}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{\partial}{\partial r}) \\ &= -\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial \theta \partial r} \\ &= -\frac{1}{r^2}\vec{e}_\theta = -\frac{1}{r}\vec{e}_{\hat{\theta}}\end{aligned} \tag{2.44}$$

To find the structure coefficients in coordinate basis we must use $[\vec{e}_{\hat{r}} \ , \ \vec{e}_{\hat{\theta}}] = -\frac{1}{r}\vec{e}_{\hat{\theta}}$.

$$[\vec{e}_{\hat{\mu}} \ , \ \vec{e}_{\hat{\nu}}] = c^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}}\vec{e}_{\hat{\rho}} \tag{2.45}$$

Using (2.44) and (2.45) we get

$$c^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = -\frac{1}{r} \tag{2.46}$$

From the definition of $c^\rho{}_{\mu\nu}$ ($[\vec{u} \, , \, \vec{v}] = -[\vec{v} \, , \, \vec{u}]$) we see that the structure coefficients are anti symmetric in their lower indices:

$$\boxed{c^\rho{}_{\mu\nu} = -c^\rho{}_{\nu\mu}} \tag{2.47}$$

$$c^{\hat{\theta}}{}_{\hat{\theta}\hat{r}} = \frac{1}{r} = -c^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} \tag{2.48}$$