Lecture 6 30.01.2018

2.2 Tensors

A 1-form-basis $\underline{\omega}^1, \ldots, \underline{\omega}^n$ is defined by:

$$\underline{\omega}^\mu(\vec{e_\nu}) = \delta^\mu{}_\nu \tag{2.49}$$

An **arbitrary** 1-form can be expressed, in terms of its components, as a linear combination of the basis forms:

$$\underline{\alpha} = \alpha_\mu \underline{\omega}^\mu \tag{2.50}$$

where α_μ are the components of $\underline{\alpha}$ in the given basis.
Using eqs.(2.49) and (2.50), we find:

$$\begin{aligned} \underline{\alpha}(\vec{e}_\nu) &= \alpha_\mu \underline{\omega}^\mu(\vec{e}_\nu) = \alpha_\mu \delta^\mu_\nu = \alpha_\nu \\ \underline{\alpha}(\vec{v}) &= \underline{\alpha}(v^\mu \vec{e}_\mu) = v^\mu \underline{\alpha}(\vec{e}_\mu) = v^\mu \alpha_\mu = v^1\alpha_1 + v^2\alpha_2 + \ldots \end{aligned} \tag{2.51}$$

We will now look at functions of multiple variables.

Definition 2.2.1 (Multilinear function, tensors)
A multilinear function is a function that is linear in all its arguments and maps one-forms and vectors into real numbers.

- A **covariant tensor** only maps vectors.
- A **contravariant tensor** only maps forms.
- A **mixed tensor** maps both vectors and forms into R.

A tensor of **rank** $\binom{N}{N'}$ maps N one-forms and N' vectors into R. It is usual to say that a tensor is of rank $(N + N')$. A one-form, for example, is a covariant tensor of rank 1:

$$\underline{\alpha}(\vec{v}) = v^\mu \alpha_\mu \tag{2.52}$$

Definition 2.2.2 (Tensor product)
The basis of a tensor R of rank q contains a **tensor product**, $\otimes$. If T and S are two tensors of rank m and n, the tensor product is defined by:

$$T \otimes S(\vec{u_1},..., \vec{u_m}, \vec{v_1},..., \vec{v_n}) \equiv T(\vec{u_1},..., \vec{u_m}) S(\vec{v_1},..., \vec{v_n}) \tag{2.53}$$

where T and S are tensors of rank m and n, respectively. $T \otimes S$ is a tensor of rank $(m+n)$.

Let $R = T \otimes S$. We then have

$$R = R_{\mu_1,...,\mu_q} \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \cdots \otimes \underline{\omega}^{\mu_q} \tag{2.54}$$

Notice that $S \otimes T \neq T \otimes S$. We get the components of a tensor (R) by using the tensor on the basis vectors:

$$R_{\mu_1,...,\mu_q} = R(\vec{e_{\mu_1}},..., \vec{e_{\mu_q}}) \tag{2.55}$$

The indices of the components of a contravariant tensor are written as upper indices, and the indices of a covariant tensor as lower indices.

Example 2.2.1 (Example of a tensor)
Let $\vec{u}$ and $\vec{v}$ be two vectors and $\underline{\alpha}$ and $\underline{\beta}$ two 1-forms.

$$\vec{u} = u^\mu \vec{e}_\mu; \quad \vec{v} = v^\mu \vec{e}_\mu; \quad \underline{\alpha} = \alpha_\mu \underline{\omega}^\mu; \quad \underline{\beta} = \beta_\mu \underline{\omega}^\mu \tag{2.56}$$

From these we can construct tensors of rank 2 through the relation $R = \vec{u} \otimes \vec{v}$ as follows: The components of R are

$$\begin{aligned} R^{\mu_1 \mu_2} &= R(\underline{\omega}^{\mu_1}, \underline{\omega}^{\mu_2}) \\ &= \vec{u} \otimes \vec{v}(\underline{\omega}^{\mu_1}, \underline{\omega}^{\mu_2}) \\ &= \vec{u}(\underline{\omega}^{\mu_1}) \vec{v}(\underline{\omega}^{\mu_2}) \\ &= u^\mu \vec{e}_\mu(\underline{\omega}^{\mu_1}) v^\nu \vec{e}_\nu(\underline{\omega}^{\mu_2}) \\ &= u^\mu \delta^{\mu_1}_\mu v^\nu \delta^{\mu_2}_\nu \\ &= u^{\mu_1} v^{\mu_2} \end{aligned} \tag{2.57}$$

2.2.1 Transformation of tensor components

We shall not limit our discussion to coordinate transformations. Instead, we will consider arbitrary transformations between bases, $\{\vec{e}_\mu\} \longrightarrow \{\vec{e}_{\mu'}\}$. The elements of transformation matrices are denoted by $M^{\mu}_{\ \mu'}$ such that

$$\vec{e}_{\mu'} = \vec{e}_\mu M^{\mu}_{\ \mu'} \quad \text{and} \quad \vec{e}_\mu = \vec{e}_{\mu'} M^{\mu'}_{\ \mu} \tag{2.58}$$

where $M^{\mu'}_{\ \mu}$ are elements of the inverse transformation matrix. Thus, it follows that

$$M^{\mu}_{\ \mu'} M^{\mu'}_{\ \nu} = \delta^{\mu}_{\ \nu} \tag{2.59}$$

If the transformation is a coordinate transformation, the elements of the matrix become

$$\boxed{M^{\mu'}_{\ \mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}} \tag{2.60}$$

2.2.2 Transformation of basis 1-forms

$$\begin{aligned} \underline{\omega}^{\mu'} &= M^{\mu'}_{\ \mu} \underline{\omega}^{\mu} \\ \underline{\omega}^{\mu} &= M^{\mu}_{\ \mu'} \underline{\omega}^{\mu'} \end{aligned} \tag{2.61}$$

The components of a tensor of higher rank transform such that every contravariant index (upper) transforms as a basis 1-form and every covariant index (lower) as a basis vector. Also, all elements of the transformation matrix are multiplied with one another.

Example **2.2.2 (A mixed tensor of rank 3)**

$$T^{\alpha'}_{\ \mu'\nu'} = M^{\alpha'}_{\ \alpha} M^{\mu}_{\ \mu'} M^{\nu}_{\ \nu'} T^{\alpha}_{\ \mu\nu} \tag{2.62}$$

The components in the **primed basis** are linear combinations of the components in the **unprimed basis**.

Tensor transformation of components means that tensors have a basis **independent** existence. That is, if a tensor has non-vanishing components in a **given basis** then it has non-vanishing components in **all bases**. This means that tensor equations have a basis independent form. **Tensor equations are invariant**. A basis transformation might result in the vanishing of one or more tensor components. Equations in **component** form may differ from one basis to another. But an equation expressed in tensor components can be transformed from one basis to another using the tensor component transformation rules. An equation that is expressed only in terms of tensor components is said to be **covariant**.

2.2.3 The metric tensor

Definition 2.2.3 (The metric tensor)
The scalar product of two vectors $\vec{u}$ and $\vec{v}$ is denoted by g($\vec{u}$,$\vec{v}$) and is defined as a symmetric linear mapping which for each pair of vectors gives a scalar g($\vec{v}$,$\vec{u}$) = g($\vec{u}$,$\vec{v}$).

The value of the scalar product g($\vec{u}$,$\vec{v}$) is given by specifying the scalar products of each pair of basis-vectors in a basis.

g is a symmetric **covariant** tensor of rank 2. This tensor is known as the **metric tensor**. The components of this tensor are

$$\boxed{g(\vec{e}_\mu, \vec{e}_\nu) = g_{\ \mu\nu}} \tag{2.63}$$

$$\vec{u} \cdot \vec{v} = g(\vec{u}, \vec{v}) = g(u^\mu \vec{e}_\mu, v^\nu \vec{e}_\nu) = u^\mu v^\nu g(\vec{e}_\mu, \vec{e}_\nu) = u^\mu u^\nu g_{\ \mu\nu} \tag{2.64}$$

Usual notation:

$$\boxed{\vec{u} \cdot \vec{v} = g_{\ \mu\nu} u^\mu v^\nu} \tag{2.65}$$

The absolute value of a vector:

$$|\vec{v}| = \sqrt{g(\vec{v}, \vec{v})} = \sqrt{|g_{\mu\nu} v^\mu v^\nu|} \tag{2.66}$$

`Example` 2.2.3 (Cartesian coordinates in a plane)

$$\begin{gathered}
\vec{e}_x \cdot \vec{e}_x = 1, \qquad \vec{e}_y \cdot \vec{e}_y = 1, \qquad \vec{e}_x \cdot \vec{e}_y = \vec{e}_y \cdot \vec{e}_x = 0 \\
g_{xx} = g_{yy} = 1, \qquad g_{xy} = g_{yx} = 0 \\
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{gathered} \tag{2.67}$$

`Example` 2.2.4 (Basis-vectors in plane polar-coordinates)

$$\vec{e}_r \cdot \vec{e}_r = 1, \quad \vec{e}_\theta \cdot \vec{e}_\theta = r^2, \quad \vec{e}_r \cdot \vec{e}_\theta = 0, \tag{2.68}$$

The metric tensor in plane polar-coordinates:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{2.69}$$