Lecture 8 060218

2.3 Forms

An **antisymmetric tensor** is a tensor whose sign changes under an arbitrary exchange of two arguments.

$$A(\cdots, \vec{u}, \cdots, \vec{v}, \cdots) = -A(\cdots, \vec{v}, \cdots, \vec{u}, \cdots) \tag{2.83}$$

The components of an antisymmetric tensor change sign under exchange of two indices.

$$A_{\cdots\mu\cdots\nu\cdots} = -A_{\cdots\nu\cdots\mu\cdots} \tag{2.84}$$

Definition 2.3.1 (p-form)
A **p-form** is defined to be an antisymmetric, covariant tensor of rank p.
An antisymmetric tensor product $\wedge$ is defined by:

$$\underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\mu_p]} \wedge \underline{\omega}^{[\nu_1} \otimes \cdots \otimes \underline{\omega}^{\nu_q]} \equiv \frac{(p+q)!}{p!q!} \underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\nu_q]} \tag{2.85}$$

where [ ] denotes antisymmetric combinations defined by:

$$\underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\mu_p]} \equiv \frac{1}{p!} \cdot \text{(the sum of terms with all possible permutations of indices with, "+" for even and "-" for odd permutations)} \tag{2.86}$$

Example 2.3.1 (antisymmetric combinations)

$$\underline{\omega}^{[\mu_1} \otimes \underline{\omega}^{\mu_2]} = \frac{1}{2}(\underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} - \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1}) \tag{2.87}$$

Example 2.3.2 (antisymmetric combinations)

$$\begin{aligned} \underline{\omega}^{[\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3]} = \\ \frac{1}{3!}(\underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3} + \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} + \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_1} \\ - \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_3} - \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1} - \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_2}) \\ = \frac{1}{3!}\epsilon_{ijk}(\underline{\omega}^{\mu_i} \otimes \underline{\omega}^{\mu_j} \otimes \underline{\omega}^{\mu_k}) \end{aligned} \tag{2.88}$$

Example 2.3.3 (A 2-form in 3-space)

$$\underline{\alpha} = \alpha_{12}\underline{\omega}^1 \otimes \underline{\omega}^2 + \alpha_{21}\underline{\omega}^2 \otimes \underline{\omega}^1 + \alpha_{13}\underline{\omega}^1 \otimes \underline{\omega}^3 + \alpha_{31}\underline{\omega}^3 \otimes \underline{\omega}^1 + \alpha_{23}\underline{\omega}^2 \otimes \underline{\omega}^3 + \alpha_{32}\underline{\omega}^3 \otimes \underline{\omega}^2 \tag{2.89}$$

Now the antisymmetry of $\underline{\alpha}$ means that

$$+\underline{\alpha}_{21} = -\underline{\alpha}_{12}; \quad +\underline{\alpha}_{31} = -\underline{\alpha}_{13}; \quad +\underline{\alpha}_{32} = -\underline{\alpha}_{23} \tag{2.90}$$

$$
\begin{aligned}
\underset{\sim}{\alpha} =& \alpha_{12}(\underset{\sim}{\omega}^1 \otimes \underset{\sim}{\omega}^2 - \underset{\sim}{\omega}^2 \otimes \underset{\sim}{\omega}^1) \\
&+ \alpha_{13}(\underset{\sim}{\omega}^1 \otimes \underset{\sim}{\omega}^3 - \underset{\sim}{\omega}^3 \otimes \underset{\sim}{\omega}^1) \\
&+ \alpha_{23}(\underset{\sim}{\omega}^2 \otimes \underset{\sim}{\omega}^3 - \underset{\sim}{\omega}^3 \otimes \underset{\sim}{\omega}^2) \\
&= \alpha_{|\mu\nu|} 2\underset{\sim}{\omega}^{[\mu} \otimes \underset{\sim}{\omega}^{\nu]}
\end{aligned}
\tag{2.91}
$$

where $|\mu\nu|$ means summation only for $\mu < \nu$ (see (Misner, Thorne and Wheeler 1973)). We now use the definition of $\wedge$ with $p = q = 1$. This gives

$$\boxed{\underset{\sim}{\alpha} = \alpha_{|\mu\nu|}\underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu}$$

We can also write

$\underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu$ is the form basis.

$$\boxed{\underset{\sim}{\alpha} = \frac{1}{2}\alpha_{\mu\nu}\underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu}$$

A tensor of rank 2 can always be split up into a symmetric and an antisymmetric part. (Note that tensors of higher rank can not be split up in this way.)

$$
\begin{aligned}
T_{\mu\nu} &= \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) + \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \\
&= A_{\mu\nu} + S_{\mu\nu}
\end{aligned}
\tag{2.92}
$$

We thus have:

$$
\begin{aligned}
S_{\mu\nu}A^{\mu\nu} &= \frac{1}{4}(T_{\mu\nu} + T_{\nu\mu})(T^{\mu\nu} - T^{\nu\mu}) \\
&= \frac{1}{4}(T_{\mu\nu}T^{\mu\nu} - T_{\mu\nu}T^{\nu\mu} + T_{\nu\mu}T^{\mu\nu} - T_{\nu\mu}T^{\nu\mu}) \\
&= 0
\end{aligned}
\tag{2.93}
$$

In general, summation over indices of a symmetric and an antisymmetric quantity vanishes. In a summation $T_{\mu\nu}A^{\mu\nu}$ where $A^{\mu\nu}$ is antisymmetric and $T_{\mu\nu}$ has no symmetry, only the antisymmetric part of $T_{\mu\nu}$ contributes. So that, in

$$\underset{\sim}{\alpha} = \frac{1}{2}\alpha_{\mu\nu}\underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu \tag{2.94}$$

only the antisymmetric elements $\alpha_{\nu\mu} = -\alpha_{\mu\nu}$, contribute to the summation. These antisymmetric elements are the **form components**

Forms are antisymmetric covariant tensors. Because of this antisymmetry a form with two identical components must be a **null form** (= zero). e.g. $\alpha_{131} = -\alpha_{131} \Rightarrow \alpha_{131} = 0$

In an n-dimensional space all p-forms with $p > n$ are null forms.